Statistical approach to brittle fracture

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A general expression for the failure probability of a brittle material is developed by using the properties of flaw size distribution and the stress necessary to fracture an inclined crack. A comparison is made with Weibull analysis and an expression for the Weibull modulus, which is known to be an empirical material constant, is related to the properties of the flaw size distribution of a material. Limitations in the application of Weibull analysis are also discussed.

1. Introduction

Brittle materials exhibit a scatter of failure strengths unlike the ductile materials where plastic deformation takes place. The mode of fracture in a homogenous brittle material depends on the stress necessary to propagate an existing critical flaw or crack in it. In certain materials flaws can be inclusions, segregations or any other centres which give rise to incompatible deformations. Therefore variable sizes, shapes and orientations (with respect to the applied load) of the flaws in a material can account for the observed scatter of fracture strengths, when nominally identical specimens are tested under nominally identical loading conditions.

A statistical method commonly used to determine the strength of brittle materials is that given by Weibull [1]. In his theory an empirical formula of the form given below is used to relate the probability of failure, \( P_f \), with stress, \( \sigma \).

\[
P_f = 1 - \exp \left( - \int \left( \frac{\sigma - \sigma_\alpha}{\sigma_o} \right) ^m \, dV \right),
\]

where \( m \) is a parameter (sometimes termed the Weibull modulus) determined experimentally, \( V \) is the volume of material, \( \sigma_\alpha \) is a normalizing factor, and \( \sigma_o \) is the stress at which there is zero probability of failure. It is important to note that \( m \) which is a material characteristic has no real relationship to the micro- or macrostructure of the material. In Weibull analysis it is assumed that fracture at the most critical flaw under a given stress distribution leads to total failure. Thus, it is based on the idea of the “weakest link of a chain” concept as opposed to the parallel concept in which the failure of “one chain” causes redistribution of load among the other “chains”, with total failure only taking place when the entire system is no longer capable of carrying the redistributed load.

In this paper a general expression for the failure probability of a brittle material is developed for a uniaxial tensile loading case by using the properties of flaw size distribution of a material and the tensile stress required to propagate a crack of a given size in a specific orientation to the applied load. This theory is then compared with Weibull analysis and an expression for the Weibull modulus, hitherto considered as an empirical constant, is related to the properties of the flaw size distribution of a material.

2. Theory

The stress necessary to propagate an inclined crack, as shown in Fig. 1, has been studied by Sih [2] and Jayatilaka et al. [3] using strain energy concepts. They showed that the initial crack growth occurs when the strain energy density of a body attains a minimum value. Using this concept, the strength, \( \sigma \), of a brittle material is given by

\[
\sigma^2 a = L(\beta, \nu)
\]

where \( a \) is the semi-crack length, \( \beta \) is the crack angle and \( \nu \) is the Poisson’s ratio of the material. \( L(\beta, \nu) \) is a function of \( \beta \) and \( \nu \). Equation 2 may be rewritten in an analytical form (see Appendix 1) for \( \nu = 0.25 \) as...
Figure 1 An inclined crack under a uniform tensile stress.

$$\sigma^2 a = \frac{1}{2} K_{IC}^2 \beta^{-1}$$

(3)

where \( K_{IC} \) is the critical stress intensity factor of the material.

Let \( f(a) \) be the probability density of the semi-crack length, where \( a > 0 \). Assuming that any crack angle is equally likely, the probability density of \( \beta \) is \( \frac{2}{\pi} \) for \( 0 \leq \beta < \frac{\pi}{2} \). The probability of failure, \( F(\sigma) \), at a stress, \( \sigma \), for one crack is given by

$$F(\sigma) = \int \int \frac{2}{\pi} f(a) da da$$

(4)

where \( L(\beta, \nu) = \frac{K_{IC}^2}{2a^2} \)

$$0 \leq \beta \leq \frac{\pi}{2}.$$  

From Equations 2 and 3,

$$L(\beta, \nu) = \frac{1}{2} K_{IC}^2 \beta^{-1}$$

(5)

Experimental results by Poloniecki [4], and Poloniecki and Wilshaw [5] suggest that \( f(a) \) can be closely fitted by the following expression (see Fig. 2).

$$f(a) = \frac{c^{n-1}}{(n-2)!} a^{-n} e^{-c/a}$$

(6)

where \( c \) is a scaling parameter and \( n \) is the rate at which the density tends to zero. It is important to note from Equation 3 that the strength is controlled by the flaws found in the “tail” of the curve. Thus, it follows that an error in the function to describe the crack size distribution for small \( a \) is not serious. Equation 4 now takes the form

$$F(\sigma) = \int \int \frac{2 c^{n-1} a^{-n} e^{-c/a}}{(n-2)!} da da$$

(7)

where \( K_{IC}^2 / 2a^2 \)

$$0 \leq \beta \leq \frac{\pi}{2}.$$  

Fig. 3 shows the limits of integration. On substituting the limits, Equation 7 takes the form

$$F(\sigma) = \int_{K_{IC}/(\pi a^2)}^{\infty} \left[ \int \frac{\pi / 2}{K_{IC}/(2a^2) \pi (n-2)!} e^{-c/a} \right] da$$

(8)

and one integration gives

$$F(\sigma) = \int_{K_{IC}/(\pi a^2)}^{\infty} \left[ 1 - \frac{K_{IC}^2}{\pi a^2} \right] \frac{c^{n-1} a^{-n} e^{-c/a}}{(n-2)!} da.$$  

(9)