A note on Pólya enumeration theory*

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The power group method of de Bruijn and Harary in enumeration under group action of mappings between finite sets is extended to include correlation of group actions on domain and range. By relaxing the restriction of weight functions to be constant over orbits, more specific results concerning the enumeration of orbits by weight are obtained.

Key words: Pólya enumeration theory—power group theorem—Burnside’s lemma—symmetry types of mappings—correlation of symmetries

This note deals with what is today called Pólya Enumeration Theory, i.e. the body of material centered about Pólya’s famous theorem [1] and its generalizations by de Bruijn [2], Harary [3, 4] and many others. More specifically, it is concerned with an extension of the power group enumeration theorem that was introduced by de Bruijn [2] and further elaborated by Harary and Palmer [3], who coined its name as well. Pólya’s theorem enumerates the orbits of mappings between finite sets with respect to a group of permutations on their domain. This setting is generalized in power group enumeration by introducing, besides the domain group, also a group of permutations on the range, which then additionally reduces the number of distinct patterns (orbits). However, this generalization is of a very special type, since the groups of domain and range act jointly but independently of each other, i.e. there is no correlation between the symmetries of domain and range. From the viewpoint of a “chemical combinatorics” it is rather more natural to consider, instead of two such permutation groups, a single point group, say, that acts on the domain and on the range simultaneously. As a consequence, if there is a non-trivial action on both, domain and range, the possibility of some correlation between these actions emerges quite naturally. The present paper

* This paper is dedicated to Professor Dr. Ernst Ruch on the occasion of this 65th birthday
proposes an extension of power group enumeration that includes correlation of symmetries. Moreover it proposes to relax the restriction on weight functions to be constant on any orbit, which is demanded by the “weighted version” of the Cauchy–Frobenius Lemma. Modern presentations of Pólya Enumeration Theory such as [6, 7, 8, 9] almost inevitably employ this version in order to derive generating functions for the numbers of orbits with various prescribed properties. While the restriction mentioned above is not operative in the case of pure domain action, in power group enumeration it has somewhat unpleasant consequences by lumping together objects that one would like to consider separately. We offer an extension of the weighted Cauchy–Frobenius Lemma that admits a much larger class of weight functions. The generating functions that are obtained in this manner provide more specific results concerning the enumeration of orbits by weight, as compared with the conventional approach.

Both these ideas of introducing correlation between group actions on domain and range, and of abandoning constant weight functions are occasionally mentioned in the literature or used for another purpose, as for instance in [10] and [11]. But to the present author’s knowledge they were never implemented in the body of power group enumeration, which then is the intention of this note.

Let us begin with a summary of the basic facts in Pólya Enumeration Theory, presented from the viewpoint of a single group acting on mappings by acting on their domain or both, on domain and range.

A finite group $G$ is said to act on a finite set $M$ if the elements of $G$ act as permutations on $M$, more explicitly, if to each $g \in G$ a permutation $\sigma_g \in \text{Sym} (M)$ is associated such that the mapping $\sigma: g \mapsto \sigma_g$ is a homomorphism from $G$ into $\text{Sym} (M)$, the symmetric group of $M$. That is

$$\sigma_g \sigma_{g'} = \sigma_{g'g} \quad \text{for any } g, g' \in G. \quad (1)$$

Synonymously, $M$ affords a permutation representation of $G$, or $M$ is a $G$-set. The action of the group $G$ induces an equivalence relation on the set $M$, $m' \sim m \iff \exists g \in G: m' = \sigma_g (m), \quad (2)$

due to which this set decomposes into orbits, i.e. equivalence classes under group action. For $m \in M$, the symbol $O_G(m)$ will denote the orbit that contains $m$, so

$$O_G(m) := \{ m' = \sigma_g (m) | g \in G \}. \quad (3)$$

The action of $G$ associates to each $m \in M$ a subgroup of $G$, its stabilizer

$$G_m := \{ g \in G | \sigma_g (m) = m \}, \quad (4)$$

which is related to the orbit $O_G(m)$ by the fact that the orbit length $|O_G(m)|$, i.e. the number of elements in the orbit, is given by the stabilizer index.

$$|O_G(m)| = \frac{|G|}{|G_m|}. \quad (5)$$

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1 Which is usually, but erroneously, attributed to Burnside, cf. [5]