NONLINEAR SPINOR FIELDS IN BIANCHI-I SPACE: EXACT SELF-CONSISTENT SOLUTIONS

Yu. P. Rybakov, B. Sakha, and G. N. Shikin

Calculations are performed to obtain exact self-consistent solutions of nonlinear spinor-field equations with self-action terms in Bianchi-I space. The latter terms are arbitrary functions of the invariant \( s = \bar{\phi} \bar{\phi} \). A detailed examination is made of equations with exponential nonlinearity, when the nonlinear term in the Lagrangian of the spinor field \( L_n = \lambda \bar{\phi}^n \). Here, \( \lambda \) is the nonlinearity parameter, \( n > 1 \). It is shown that these equations have finite solutions and solutions that are singular at the initial moment of time. The singularity is absent in the case of solutions that describe systems of fields for which the energy dominance condition is violated. It is further shown that if the mass parameter \( m \neq 0 \) in the spinor-field equation, expansion of Bianchi-I space becomes isotropic as \( t \to \infty \). This does not occur when \( m = 0 \). Specific examples of solutions of linear and nonlinear spinor-field equations are presented.

The Lagrangian of a self-consistent system of spinor and gravitational fields has the form

\[
L = R/2\kappa + (i/2) (\bar{\psi} \gamma^\mu \gamma_\nu \bar{\psi} - \gamma_\nu \gamma^\mu \bar{\psi} \gamma_\nu \bar{\psi}) - m\bar{\psi} \psi + L_N. \tag{1}
\]

where \( R \) is scalar curvature; \( \kappa \) is Einstein’s gravitational constant; the function \( L_N = F(S) \), \( S = \bar{\psi} \bar{\psi} \), arbitrarily dependent on \( S \), determines the nonlinear term in the Lagrangian of the spinor field.

The metric of the Biachi-I state is chosen in the form [1]

\[
ds^2 = dt^2 - a^2(t) dx^2 - b^2(t) dy^2 - c^2(t) dz^2. \tag{2}
\]

We use Lagrangian (1) to obtain the Einstein equations, the equations of the spinor field, and the components of the energy-momentum tensor of the spinor field. We use the Einstein equations in the form [1]:

\[
\dot{a}/a + (\dot{a}/a) (b/b + \dot{c}/c) = -\kappa (T^1_1 - T/2), \tag{3}
\]

\[
\dot{b}/b + (\dot{b}/b) (\dot{c}/c + a/a) = -\kappa (T^2_2 - T/2), \tag{4}
\]

\[
\dot{c}/c + (\dot{c}/c) (\dot{a}/a + \dot{b}/b) = -\kappa (T^3_3 - T/2), \tag{5}
\]

\[
\dot{a}/a + \dot{b}/b + \dot{c}/c = -\kappa (T^0_0 - T/2), \tag{6}
\]

where the dots denote differentiation with respect to \( t \).

The equations of a nonlinear spinor field and the components of its energy—momentum tensor are written thusly:

\[
(i \gamma^\mu \gamma_\nu - m) \bar{\psi} + L'_N \bar{\psi} = 0, \quad L'_N \bar{\psi} = \partial L_N / \partial \bar{\psi}. \tag{7}
\]
\[ T^\mu_\nu = \left( i/2 \right) (\bar{\psi} \gamma^\mu \gamma^\nu \psi - \gamma^\mu \bar{\psi} \gamma^\nu \psi) + \gamma_\mu \left( (1/2) \left( \frac{\partial L_N}{\partial \bar{\psi}} + \frac{\partial L_N}{\partial \psi} \right) - L_N \right). \]  

(8)

where \( \nabla_\mu \) is a covariant derivative having the form [2]

\[ \nabla_\mu \psi = \partial_\mu \psi - \Gamma_\mu \psi. \]

(9)

\( \Gamma_\mu(x) \) is an affine spinor matrix. The matrices \( \Gamma_\mu(x) \) are determined as follows in metric (5). With use of the equalities

\[ g_{\mu \nu}(x) = e^\alpha_a(x) e^\beta_b(x) \eta_{\alpha \beta}, \quad \gamma_a(x) = e^\alpha_a(x) \gamma_a, \]

where \( \eta_{ab} = \text{diag}(1, -1, -1, -1) \), \( \gamma_a \) is the Dirac matrix of flat space-time, and \( e^\alpha_a(x) \) is a set of tetrad 4-vectors, we obtain:

\[ \gamma^0 = \bar{\gamma}^0, \quad \gamma^1 = \bar{\gamma}^1/a(t), \quad \gamma^2 = \bar{\gamma}^2/b(t), \quad \gamma^3 = \bar{\gamma}^3/c(t). \]

Matrix \( \Gamma_\mu(x) \) is determined from the equality

\[ \Gamma_\mu(x) = (1/4) g_{\mu \nu}(x) (\partial_\nu e^\rho_b - \Gamma^\rho_{\nu b}) \gamma^\rho \gamma^\nu, \]

from which we obtain

\[ \Gamma_0 = 0, \quad \Gamma_1 = \dot{a}(t) \bar{\gamma}^1 \gamma^0/2, \quad \Gamma_2 = \dot{b}(t) \bar{\gamma}^2 \gamma^0/2, \quad \Gamma_3 = \dot{c}(t) \bar{\gamma}^3 \gamma^0/2. \]

(10)

We choose to use the matrices of flat space-time in the form given in [3].

Let us examine the solutions of spinor equations for which the field function depends only on \( t \): \( \psi = \psi(t) \). In this case, Eq. (7) is written as follows with allowance for (9) and (10):

\[ (i \bar{\psi} \gamma^0 (\partial_\tau + i/2 \tau) - (m - L_N)) \gamma \psi = 0, \quad \tau(t) = a(t) b(t) c(t). \]

(11)

We obtain the below system of equations from (11) for the components \( \psi_a = \psi_a(t), \alpha = 1, 2, 3, 4 \):

\[ \dot{V}_\alpha + \left( i V_i/2 \tau \right) + i (m - F_i) V_\alpha = 0; \quad \dot{V}_i + \left( i V_i/2 \tau \right) - i (m - F_i) V_\tau = 0; \]

(12)

where \( F_1 = \frac{dF}{d\tau} \). We use (12) to find an equation for the invariant function

\[ S = \bar{\psi} \psi \dot{V}_1 V_1 + \dot{V}_2 V_2 - \dot{V}_3 V_3 - \dot{V}_4 V_4; \]

\[ \ddot{S} + (\dot{\tau}/\tau) S = 0, \]

(13)

from which we obtain

\[ S(t) = C_0/\tau(t), \quad C_0 = \text{const}. \]

(14)

Since \( F \) depends only on \( S \) in the given case, it follows from (14) that \( F(S) \) and \( F_1(S) \) are functions of the product \( \tau(t) = a(t)b(t)c(t) \). With allowance for this integration of system (12) leads to the equalities

\[ V_\alpha(t) = (C_\tau / V_\tau) e^{-i(m - F_\alpha)} dt, \quad V_i(t) = (C_l / V_\tau) e^{i(m - F_i)} dt, \]

(15)

where \( C_\tau \) and \( C_l \) are constants of integration.

Inserting (15) into (8), we obtain the following expressions for the components of the energy-momentum tensor of the spinor field:

\[ T^0_\mu = (i/2) N + R, \quad T^1_\mu = T^2_\mu = T^3_\mu = R, \quad T^\tau_\mu = (i/2) N + 4R, \]

(16)