We discuss the solutions of quantum problems which are nonlinear in the classical limit. It is shown that in this case it is necessary to solve the corresponding nonlinear classical problem and study its bifurcation properties.

Dynamical chaos in quantum systems (quantum chaos) has been studied by a large number of physicists [1-7]. It has been shown that quantum chaos is possible only in the quasiclassical region, for example in the case of quantum states near the dissociation limit in molecules or near the ionization limit in atoms and molecules. However, the quantum equations of motion are linear and hence they cannot contain dynamical chaos. In the present paper we discuss the solution of quantum problems which are nonlinear in the classical limit. It will be show that in solving the quantum problem in this case it is necessary to use the solution of the nonlinear classical problem forming the classical limit of the quantum problem.

1. We note first that here we are not interested in the usual field nonlinearities considered, for example, in quantum optics. This is because field nonlinearities cannot determine the bifurcation properties of the quantum equations of motion since the latter are linear in the wave function \( \psi \) or density matrix \( \rho \). In other words, dynamical chaos, which is intimately connected with the bifurcation properties of the corresponding equations, is not determined by nonlinearities in the fields in quantum equations of motion linear in \( \psi \) or \( \rho \).

Obviously when we speak of linear quantum equations we mean the fundamental equations such as the Schrödinger equation or the Neuman background equation. Phenomenological theories can be nonlinear in the wave function, such as the Ginzburg–Landau theory of superconductivity (see [8]). Interesting physical and methodological aspects of the Ginzburg–Landau theory will be considered at the end of the present paper.

2. Let the Hamiltonian of the problem have the form

\[
H(x; \mu) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + U(x; \mu),
\]

where \( U(x; \mu) \) is the potential function and \( \mu \) is a control parameter (or a set of control parameters). The quantum problem can be written in the form

\[
\dot{\psi}(x, t; \mu) = iH(x; \mu)\psi(x, t; \mu).
\]

The classical limit of the quantum problem (1) will be defined by the equation

\[
\ddot{x} = F(x; \mu),
\]

where the operator \( F(x; \mu) \) is related to the potential function \( U(x; \mu) \) by the equation

\[
F(x; \mu) = -\frac{\partial U(x; \mu)}{\partial x}.
\]

If \( F(x; \mu) \) is a linear operator then the corresponding quantum and classical problems are completely autonomous. However if \( F(x; \mu) \) is a nonlinear operator then to solve the quantum problem it is necessary to turn to its classical limit (3).
We consider as an example a potential of the form
\[ U(x; \mu_1, \mu_2) = \mu_1 x^2 + \mu_2 x^4. \] (5)

Here \( \mu_1 \) and \( \mu_2 \) play the role of control parameters. We consider two cases.

a) \( \mu_1 > 0, \mu_2 > 0 \). In this case the shape of the potential (5) is shown in Fig. 1. It is evident that in this case the quantum problem has discrete states
\[ \psi(x, t; \mu_1, \mu_2; n), \] (6)
where \( n \) is the quantum number. A solution of the type (6) exists far from the bifurcation points of (3), i.e., when \( \mu < \mu_c \) (\( \mu = (\mu_1, \mu_2) \)). Here \( \mu_c \) corresponds to values of the control parameters for which bifurcation occurs in (3) with a nonlinear operator \( F(x, \mu) \). In other words, in this region \( \mu < \mu_c \) a smooth change in the control parameters does not change the general nature of the solution (6). In this sense this case differs only slightly from the case when \( F(x; \mu) \) is a linear operator.

b) \( \mu_1 > 0, \mu_2 < 0 \) (\( \mu > \mu_c \)). The possible shapes of the potential for this case are shown in Fig. 2. Local maxima of the potential (5) occurs at the points \( x_{1,2} = \pm (\mu_1/2|\mu_2|) \). The solution of the quantum problem in this case involves the continuous spectrum and so it changes in a fundamental way when we go from the region \( \mu < \mu_c \) into the region \( \mu > \mu_c \). It is evident that it is impossible to transform the solution (6) for case a into the solution for case b by a smooth change of the control parameters from \( \mu < \mu_c \) to \( \mu > \mu_c \). Hence, in the solution of a quantum problem nonlinear in the classical limit it is necessary to first consider this classical limit and study its bifurcation points and only then consider the solution of the quantum problem. A bifurcation point of the nonlinear classical limit represents for the quantum problem a point for which the solution is fundamentally different on the two sides of the point. For example, on one side it may describe bound states and on the other scattering states.

3. The bifurcation point \( \mu = \mu_c \) of the nonlinear classical limit characterizes a qualitative or topological change in the system. For example, this point corresponds to a saddle point, some of which may be unstable or there can be a change in stability or limit cycles can appear or disappear. The presence of an instability can bring the system to a state of dynamical chaos. However, in the quantum case, because of the condition \( \langle \psi | \psi \rangle = 1 \) (or \( \text{tr}\rho = 1 \)) the modulus \( |\psi| \) cannot become unstable (the linearity of the quantum equation of motion for \( \psi \) prevents this unphysical behavior). However, in the case of the nonlinear phenomenological Ginzburg–Landau equation instability and dynamical chaos are possible, not for \( |\psi| \), rather for the phase of the wave function. In the stationary case we obtain from the Ginzburg–Landau equation a nonlinear equation of the sine-Gordon form for the phase difference \( \phi \) of the wave functions across a Josephson junction
\[ \frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\lambda^2} \sin \phi, \] (7)