MATHEMATICAL DESCRIPTION OF ROTARY MOTION

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A mathematical description of the rotary motion of a solid is proposed on the basis of a multiplicative integral. As a preliminary, the representation of the angular-velocity vector in spherical coordinate systems and the conversion matrix corresponding to rotation of the coordinate system by a specified angle around an axis with specified orientation are discussed.

Rotary motion of a solid constitutes a complex problem [1]. Given that the general solution of this problem has not been found, it is of interest to develop new methods of describing rotary motion. These methods may prove useful in analyzing individual aspects of the given problem.

To formulate the initial hypotheses, let us consider the representation of the angular-velocity vector in a spherical coordinate system.

Some features of a spherical coordinate system in comparison with a Cartesian system will first be noted. In a three-dimensional Cartesian system, as is well known, some vectorial physical quantity is uniquely determined by specifying three projections of the vector onto coordinate axes directed along three mutually orthogonal unit vectors \( i, j, k \). Each projection is measured in the same physical units as the length of the vector.

The situation is different in a spherical coordinate system. In this case, regardless of the dimensionality of the vectorial physical quantity, the direction of the vector is specified by two dimensionless quantities: the polar angle \( \theta \) and azimuthal angle \( \varphi \). The third parameter determines the length of the vector and is measured in units corresponding to the vectorial physical quantity. This third parameter must be of the same dimensionality as the vectorial physical quantity.

The most convenient and widely used physical quantity characterizing the rotary motion of a solid is the angular-velocity vector [1]

\[
\omega = \frac{d\varphi}{dt}.
\]

In the spherical coordinate system, in accordance with the foregoing, the direction of the given vector must be specified by means of the angles \( \theta \) and \( \varphi \) and the length of the vector by means of the quantity \( |\omega| = \phi \) (Fig. 1).

The relations between these three quantities and the projections of the vector \( \omega \) onto the orthogonal axes of the Cartesian coordinate system are as follows

\[
\omega_x = \dot{\varphi} \cdot \sin \theta \cos \varphi,
\]

\[
\omega_y = \dot{\varphi} \cdot \sin \theta \sin \varphi,
\]

\[
\omega_z = \dot{\varphi} \cdot \cos \theta.
\]

Clearly, these relations are completely analogous with those between the coordinates of a point in Cartesian and spherical coordinate systems.
The three quantities \( \theta, \psi, \) and \( \varphi \) are independent and uniquely characterize the spatial position of a solid body with one fixed point. Since the set of generalized coordinates must satisfy these two requirements, the angles \( \theta, \psi, \) and \( \varphi \) may be chosen as generalized coordinates.

In the general case of free rotary motion of a solid around its center of mass in an inertial reference frame, change in the direction of vector \( \omega \) in the course of motion is possible. In this case, \( \theta \) and \( \psi \) will also vary over time. Thus, the derivatives \( \dot{\theta}, \dot{\psi}, \) and \( \dot{\varphi} \) may be regarded as generalized velocities.

The Euler angles and their time derivatives are also often used as generalized coordinates and velocities. As is well known, the relation between these quantities and the projections of \( \omega \) onto the orthogonal axes of the fixed Cartesian coordinate system is as follows:

\[
\begin{align*}
\omega_x &= \dot{\theta}' \cos \psi' + \dot{\psi}' \sin \theta' \sin \psi', \\
\omega_y &= \dot{\theta}' \sin \psi' - \dot{\psi}' \sin \theta' \cos \psi', \\
\omega_z &= \dot{\psi}' + \dot{\theta}' \cos \psi'.
\end{align*}
\]

Despite the considerable similarity between the definitions of \( \theta, \psi, \) and \( \varphi \) and \( \theta', \psi', \) and \( \varphi' \), the use of these two sets of generalized coordinates and velocities in the Lagrangian equations leads to results of significantly different form.

As an illustration, consider a simple example: the rotary motion of a free spherical gyroscope around its center of mass relative to an inertial reference frame. The Lagrangian function corresponding to free rotary motion of a solid may be written in the form

\[
L = \tilde{\omega} J \omega / 2,
\]

where \( J \) is the inertia tensor; the tilde denotes transposition.

In this particular case, considering a spherical gyroscope, \( J \) may be regarded as a scalar, and the Lagrangian function in Cartesian coordinates may be expressed in the form

\[
L = \frac{1}{2} \omega^T \tilde{J} \omega,
\]