ONSAGER PROBLEM: SOLVING THE UNSOLVED
ISING PROBLEMS

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It is shown in the present work, which is methodological in character, that the system of equations for the Ising-model correlation functions, which Doman and Ter Haar tried to render internally closed, remains open even after the use of the additional physical postulate that the correlations are damped over time.

INTRODUCTION

Of the Ising problems (IP), only the one-dimensional (fieldless and field versions) and two-dimensional (fieldless) problems have been accurately solved as yet. The accurate solution of the two-dimensional problem was first given by Onsager fifty years ago, in 1944, for a square lattice with no external magnetic field [1]. His solution, which was extremely complex both mathematically and in technical terms, consisted in the direct calculation of the statistical sum of the square lattice in terms of the spur of some matrix (the transition matrix). Although considerably simpler means of solving this IP were found thanks to the efforts of a number of authors (notably Kaufman [2] and, in particular, Vdovichenko [3]), they are still too complex and cannot be generalized to the unsolved IP [4]. Feynman [5] characterized the solution of the unsolved IP as the Onsager problem (OP). Moreover, there is extreme skepticism at present regarding the possibility of direct solution of the two-dimensional field IP and, even more so, the three-dimensional IP, although attempts to find accurate analytical solutions of these IP continue, of course.

Basically, two approaches to the search for an accurate solution of the OP have been adopted: 1) Onsager’s original approach, the calculation of the statistical sum [6]; 2) construction of the equations of motion (EM) for two-time temperature Green’s functions (GF), leading to a system of algebraic equations for the Ising-model GF [7]. By comparison with the first, the second approach is extremely simple both mathematically and in technical terms, but it still leads to an infinite system of algebraic equations for the correlation functions (CF). In 1962, it appeared that Doman and Ter Haar [8] had shown the possibility of obtaining the accurate sum of the GF equations analytically and thereby obtaining the accurate solution of the Ising model. However, in March 1963, Callen [9] showed that this complete summation of the equations for the GF does not lead to solution of the IP and in fact is equivalent only to some trivial identity. No additional assumptions (external to the GF method) were used in this work.

In the present work, another formal possibility of accurate solution of the OP is shown, but this too leads only to a contentless identity. Why should we publish this finding? Its publication may prove useful in at least three ways. First, this approach is associated with the introduction of an external physical assumption and is therefore of purely methodological interest. Second, it will undoubtedly be of prophylactic value. Third, these accurate and nontrivial manipulations (both those of Doman and Ter Haar and those used in the present work) may ultimately lead to accurate manipulations that may have a content and hence lead finally to a solution of the Onsager problem. In the present work, as in [8], the solution is sought by obtaining a closed system of algebraic equations, although not in terms of the GF but directly in terms of the CF. The point is that the free energy (which is what the statistical sum gives) of this system may always be expressed in terms of its internal energy [10, 11]

\[ \beta F(\beta) = \beta U(\infty) + \int_{\beta'}^\infty [U(\infty) - U(\beta')] \, d\beta' \] (1)
(the correctness of this formula is discussed in [12]). To calculate the internal energy in all the IP, only two of the CF must be known; see Eq. (6). The approach to obtaining a closed system of algebraic equations for the CF is based on the external physical postulate of damping of the correlations over time [13, 14]

$$\lim(|t| \to \infty) \{<\hat{A}_2(t)\hat{A}_1>| = <\hat{A}_1| <\hat{A}_2>|$$

(2)

(this is what permits closure of the system of equations) and on the wrapping-unwrapping method [15, 16], which, even in comparison with the GF method, is an extremely simple approach to deriving a system of algebraic equations for the CF. The new physical constraint in Eq. (2) was apparently first introduced in [13]. Some of its physical consequences were discussed in [14]. In the present work, this postulate is not discussed; it is simply used. However, it is helpful, as a preliminary step, to explain the reason why the system of equations obtained in the wrapping-unwrapping method is not closed (it is essentially the same as in the GF method).

1. CALCULATION OF THE ISING-MODEL CF

In the case of lattice points with spin 1/2, the spin-deviation operators in the Ising model

$$\hat{H} = -\frac{1}{2} \sum_{\langle i, j \rangle} \hat{S}_i \cdot \hat{S}_j - \frac{1}{2} J \sum_{\langle i, j \rangle} \hat{S}_i \cdot \hat{S}_j$$

(3)

are expediently replaced by creation and annihilation operators of hypothetical fermions at the lattice points

$$\hat{S}_i^+ = (1/2) \hat{a}_i; \hat{S}_i^- = (1/2) \hat{a}_i^\dagger.$$ (4)

Then the initial Hamiltonian in Eq. (3) takes the form

$$\hat{H} = -\frac{1}{2} \sum_{\langle i, j \rangle} \hat{a}_i \cdot \hat{a}_j - \frac{1}{2} J \sum_{\langle i, j \rangle} \hat{a}_i \cdot \hat{a}_j + H; \quad \hat{B} = \lambda B - J d; \quad \hat{n}_i = \sum_{\langle i, j \rangle} \hat{a}_i \hat{a}_j.$$ (5)

Here, as in Eq. (3), the subscript j runs over the z = 2d lattice points adjacent to point f; d = 1, 2, 3 is the dimensionality of the lattice (linear, d = 1; quadratic, d = 2; spatially coordinated, d = 3). The specific internal energy of the lattice (i.e., its internal energy per lattice point) is

$$<\hat{H}/N> = U(\beta) = \left\{-\left[\frac{2\lambda B + Jd}{4}\right] + \hat{B} \hat{n}_i \right\} - \frac{1}{2} J \hat{n}_i \hat{n}_j.$$

Thus, to calculate the internal energy of the system, with a view to using Eq. (1), it is sufficient to known CF of only two types: $<\hat{n}_f> = <\hat{n}> = n$ and $<\hat{n}_f \hat{n}_j>$. If $U(\beta)$ is known, it is simple to calculate its specific heat at constant volume (i.e., when $B = \text{const}$)

$$C_B = \left(\frac{\partial^2 U}{\partial T^2}\right) U(\beta) = -\kappa \beta^2 \left(\frac{\partial^2 U}{\partial \beta^2}\right) U(\beta).$$ (7)

The specific magnetization of the lattice is

$$\lambda <\hat{S}_i> = \lambda <(1/2) \hat{a}_i> = m(\beta; B).$$ (8)

The specific static (i.e., when $B = \text{const}$) susceptibility

$$X = \left(\frac{\partial}{\partial B}\right) m(\beta; B = 0) = -\lim_{B \to 0} \left[\kappa \left(\frac{\partial}{\partial B}\right) \right] \hat{n}_i.$$ (9)

The dimensionless phase-transition (PT) temperature is determined from the condition

$$m(\beta_c; B = 0) = 0.$$ (10)