QUASICLASSICAL TRAJECTORY-COHERENT STATES FOR A CALDIROLA–KANAI OSCILLATOR

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Quasiclassical trajectory-coherent states (TCS's) are constructed for an oscillator with Caldirola–Kanai Hamiltonian, having the following fundamental property: the quantum mechanical means of the coordinate and momentum operators calculated over these states are exact solutions of the corresponding classical system Hamiltonian. The properties of the solutions so constructed are studied, as is the question of minimization of the TCS Heisenberg uncertainty relationships.

INTRODUCTION

The current physics literature is much concerned with study of the properties of coherent states and their application to practical quantum mechanics problems. A large number of studies [1, 2] have recently offered various methods for coherent state construction and use in concrete problems. In particular, Nieto and Simmons [3] constructed coherent states of a variable frequency harmonic oscillator. The states constructed in [4] have practically all the basic properties of coherent states, namely, they are eigenstates for the generation–annihilation operators and can be obtained from the ground state by a unitary operator; they are normalized, but not orthogonal; and the coherent states minimize the Heisenberg uncertainty relationships.

Below we will use Maslov's theory [5, 6] to construct quasiclassical TCS's of a non-relativistic particle [7, 8] with Caldirola–Kanai Hamiltonian [9–11]. The states thus constructed have the following basic property: the quantum mechanical means of the coordinate and momentum operators calculated over those states yield the exact classical particle trajectory. It will be shown that the solutions of the Schrödinger equation for the Caldirola–Kanai Hamiltonian are exact. The question of minimization of the uncertainty relationships will also be considered. It will be shown that in the particular case of formulation of a special Cauchy problem for the Schrödinger equation (which corresponds to concrete choice of coordinate dispersion for a Gaussian packet at the initial moment of time) the results obtained coincide with those of [11]. It develops that for the condition $1/4\gamma^2 - \omega_0^2 < 0$, where $\gamma$ is the damping coefficient and $\omega_0$ is the natural frequency of the oscillator, the time function $g(t)$ defining the moment of minimization ($g(0) = 0$) is periodic, and minimization occurs after definite time intervals, while for the case $1/4\gamma^2 - \omega_0^2 > 0$ $g(t)$ has a unique minimum for $t > 0$ at the time $t_0$, which is determined by the problem parameters and the choice of initial coordinate dispersion.

1. QUASICLASSICAL TCS

We will consider a nonsteady one-dimensional Schrödinger equation with Caldirola–Kanai Hamiltonian

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi,$$  

(1)

where

$$\hat{H} = \exp (-\gamma t) (2m)^{-1/2} \hat{p}^2 + \frac{1}{2} \exp (\gamma t) m \omega_0^2 x^2, \quad \hat{p} = -i\hbar \hat{\partial}_x,$$

corresponds to the classical Hamiltonian function

$H(x, p, t) = e^{\gamma t} (2m)^{-1} p^2 + \frac{1}{2} e^{\gamma t} m \omega_0^2 x^2$  

(2)

with attenuation coefficient $\gamma$ and natural oscillator frequency $\omega_0$. The problem's Lagrangian and the classical energy have the following form [11]:

$$L = \exp(\gamma t) \left( \frac{1}{2} m x^2 - \frac{1}{2} m \omega_0^2 x^2 \right), \quad E = \exp(-2\gamma t) (2m)^{-1} p^2 + \frac{1}{2} m \omega_0^2 x^2.$$  

(3)

We will formulate the Cauchy problem for Eq. (1), choosing the initial state $\Psi|_{t=0}$ in the form of a Gaussian packet with coordinate dispersion $\hbar/\text{Im}b$, where $b$ is a complex parameter satisfying the requirement $\text{Im} b > 0$.

As was shown in [7, 8] construction of the vacuum TCS is determined by solutions of the classical system Hamiltonian

$$x(t) = \dot{H}(x, p, t), \quad \dot{p}(t) = -\partial_x H(x, p, t)$$  

(4)

and a variation system — linearizations of the system Hamiltonian in the vicinity of the classical trajectory $x(t), p(t)$

$$\dot{\omega}(t) = -H_{xp}(t) \omega(t) - H_{xx}(t) z(t), \quad \omega(0) = b,$$

$$\dot{z}(t) = H_{pp}(t) \omega(t) + H_{px}(t) z(t), \quad z(0) = 1.$$  

(5)

where

$$H_{xp}(t) = \partial_x \partial_p H(x, p, t) \big|_{x=x(t), p=p(t)}, \quad H_{xx}(t) = \partial_x^2 H(x, p, t) \big|_{x=x(t), p=p(t)},$$

$$H_{pp}(t) = \partial_p^2 H(x, p, t) \big|_{x=x(t), p=p(t)}.$$  

The main term of the asymptotic expansion of the vacuum TCS has the form [7, 8]

$$\Psi_0(x, t, \hbar) = N \Phi(t) \exp\{i \hbar^{-1} S(x, t)\}.$$  

(6)

Here $N = (\text{Im}b(\pi\hbar)^{-1})^{-1/2}$, $\Phi(t) = (\varphi(t))^{-1/2}$,

$$S(x, t) = \int_0^t \left( \dot{x}(t) p(t) - H(x(t), p(t), t) \right) dt + p(t) (x - x(t)) + \frac{1}{2} \omega(t) z^{-1}(t) (x - x(t))^2,$$

$$\Psi_0(x, t, \hbar)|_{t=0} = N \exp\left\{i \hbar^{-1} (p_0(x - x_0) + \frac{b}{2} (x - x_0)^2) \right\}.$$  

where $x_0 = x(t)|_{t=0}$, $p_0 = p(t)|_{t=0}$. Note that in the case considered here in view of the quadratic nature of the Hamiltonian with respect to coordinates and momenta function (6) satisfies the Schrödinger equation exactly.

For the Hamiltonian of Eq. (2) the solutions of system (4), (5) have the following form:

$$x(t) = \text{ch} \omega t \exp \left( -\frac{1}{2} \gamma t \right), \quad p(t) = m \left( \omega \sinh \omega t - \frac{1}{2} \gamma \cosh \omega t \right) \exp \left( \frac{1}{2} \gamma t \right),$$

$$\omega(t) = m \exp \left( \frac{1}{2} \gamma t \right) \left( \delta \omega - \frac{1}{2} \gamma \right) \cosh \omega t + \left( \omega - \frac{1}{2} \gamma \delta \right) \sinh \omega t,$$

$$z(t) = e^{\gamma t} \left( \cosh \omega t + \delta \sinh \omega t \right),$$

$$\delta = (m \omega)^{-1} \left( b + \frac{1}{2} \gamma m \right), \quad \omega^2 = \frac{1}{4} \gamma^2 - \omega_0^2.$$  

(7)