D’alambert’s equation is used as an example to study the possibilities of a new method of exactly integrating systems of linear differential equations — the method of noncommutative integration (NI). The results confirm that use of the NI is equivalent to complete separation of the variables in the case of four-dimensional subalgebras of conformal algebra. However, the method does simplify determination of the exact solution in this instance.

A new method was recently published [1, 2] for the exact solution of linear differential equations (LDIs) on the basis of noncommutative subsets of conformal algebra. The proposed method makes it possible to construct a basis in the solution space without resorting to separation of variables — the main technique used to solve LDIs.

Here, we use the method described in [1, 2] to integrate d’Alembert’s equation:

$$\nabla^2 \Psi(x) = 0.$$  \hspace{1cm} (1)

The problem of the integration of (1) has long been studied by different authors (see [3], for example) — including the authors of this article [4] — by the method of separation of variables. It is interesting to examine aspects of the noncommutative integration (NI) of Eq. (1) and see if the given method leads to solutions with nonseparated variables.

We will examine the case of the NI of Eq. (1) when the noncommutative set forms a subalgebra of the conformal algebra of a group of invariants of Eq. (1) that is isomorphic so(2, 4). We recall [5] that in this case Eq. (1) is noncommutatively integrable within Minkowski space $R^{1,3}$ if its symmetry algebra satisfies the condition

$$\dim L + \text{ind} L = 2m - 1.$$  \hspace{1cm} (2)

For the NI of d’Alembert’s equation, we use the method proposed by A. V. Shapovalov and I. V. Shirokov [1, 2] for solving systems of equations that are integrable in a noncommutative sense.

We will examine subalgebras of a conformal algebra of dimensionality $\dim G = 1$. All possible algebras of the conformal algebra were found in [6]. In accordance with condition (2), the system will be noncommutatively integrable if

$$\dim L = 2.$$

We thus confront the problem of the classification of four-dimensional algebras. We also note that subalgebras containing either three mutually commutative operators or a local Casimir operator and two mutually commutative operators (satisfying certain algebraic conditions [7]) are integrated by the method of separation of variables (there being a complete set of three mutually commutative operators).

We choose the following as the basis of the conformal algebra

with the commutative relations

\[
\begin{align*}
[L_{ij}, L_{kl}] &= \eta_{il} L_{jl} + \eta_{jk} L_{il} - \eta_{ik} L_{jl} - \eta_{jl} L_{ik}, \\
[L_{ij}, K_i] &= \eta_{ij} K_j - \eta_{ij} K_i, \\
[L_{ij}, p_k] &= \eta_{jk} p_i - \eta_{ik} p_j, \\
[K_i, p_j] &= -2 L_{ij} + \eta_{ij} D),
\end{align*}
\]

\[ \eta_{ij} = \text{diag}(1, -1, -1, -1). \]

The basis can be realized by means of the operators

\[
\begin{align*}
p_i &= \frac{\partial}{\partial x^i}, \\
L_{ij} &= x_i p_j - x_j p_i, \\
D &= x^i p_i + 1, \\
K_i &= 2x_i D - (x_i x^j) p_j, \\
(x_i \star x_j) &= x_i^2 - x_j^2 - x_i^2 - x_j^2.
\end{align*}
\]

The results of the classification are shown in Appendix A.

An analysis of the subalgebras that satisfy the NI condition shows that all of them contain complete sets consisting of three mutually commutative operators (taking into account the Casimir operator). The following theorem is thus valid.

**Theorem 1.** All four-dimensional subalgebras of a conformal algebra that satisfy condition (2) are also integrable in the commutative sense (method of separation of variables).

However, the method of noncommutative integration might also prove useful in this case: in subalgebras not containing three mutually commutative operators, there is no need to search for a Casimir operator and thus solve a second-order equation.

To illustrate, let us present one example. We take the subalgebra

\[ <X_1, X_2, X_3, X_4> = <D, K_0 - p_0, K_0 + p_0, L_{12} >. \]

The subalgebra has the local Casimir operator

\[ K(x) = 4x^2 + x^2 - x^2. \]

Its \( \lambda \)-representation has the form

\[
\begin{align*}
l_1 &= \partial_x, \\
l_2 &= (e^x - e^{-x}) \partial_x - x^2 (e^x + e^{-x}), \\
l_3 &= -(e^x + e^{-x}) \partial_x + x^2 (e^x - e^{-x}), \\
l_4 &= I.
\end{align*}
\]

The Casimir operator will obviously simply be a constant in the \( \lambda \)-representation. Solution of the system

\[ (X_i - l_i) \Psi(x, \lambda) = 0 \]

leads to the solution of the equation

\[
U^2 (U + 1) \frac{\partial^4 \Phi(U)}{\partial U^2} + U \left( U \left( R + \frac{5}{2} \right) + 1 \right) \frac{\partial^2 \Phi(U)}{\partial U} + \frac{1}{4} (\left( R + 1 \right) (R + 2) + 4 \lambda) \Phi(U) = 0.
\]

Finally, the basis of the solution has the form

210