AXIALLY SYMMETRIC FIELDS IN MULTIDIMENSIONAL GRAVITATION

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The equations of multidimensional gravitation are examined for the case of axial symmetry. Exact solutions are obtained in the static and steady-state cases. Solutions that become spherically symmetric are identified.

INTRODUCTION

The study of multidimensional models of gravitation and cosmology is important from the viewpoint of study of the properties of unified models of fundamental physical interactions occurring at high energies [1]. The known regions of application are the early universe and later stages of gravitational collapse. Exact multidimensional models were constructed and analyzed in [2] along with spherically symmetric solutions [3]. In this investigation, we examine multidimensional axially symmetric fields in the static and steady-state cases — which is very important both for problems of collapse and for the string problem.

We will examine a Riemann space of 4 + N dimensions having the form

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu + e^{2\phi(x)} g_{mn}(y) dy^m dy^n,$$

(1)

where the Greek indices take values of 0, 1, 2, and 3 and the Roman indices take values of m, n = 4, ..., N + 3.

As was shown in [3], reduction to four dimensions for a conformally Ricci-two-dimensional auxiliary space leads to the Lagrangian density

$$L = \left[ e^{2\phi} R + \left( N^2 - 1 \right) e^{-2\phi} g_{\mu\nu} \phi_{,\mu} \phi_{,\nu} + 2\phi \phi \phi \phi_{,\mu} \phi_{,\nu} + 2 \phi \phi L_m \right] \sqrt{-g},$$

(2)

which corresponds to the equations of motion of a test body of mass m, these equations following from the variation of action

$$S = -mc\int q ds,$$

(3)

This means that test bodies in the metric $g_{\mu\nu}$ do not move along geodesics and that their effective masses are functions of a scalar field. We change over to a metric in which the bodies do move along geodesics by resorting to the conformal transformation

$$g_{\alpha\beta} = q^{-2} g_{\alpha\beta}^*,$$

(4)

Here, the Lagrangian density becomes

$$L^* = \left[ e^{-4\phi} R^* + \left( N^2 - 1 \right) e^{-2\phi} g^{*\mu\nu} \phi_{,\mu} \phi_{,\nu} + 2 \phi \phi L_m^* \right] \sqrt{-g^*}.$$

(5)

The conformal transformation


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reduces Lagrangian density (2) to quasi-Einsteinian form

\[ \bar{L} = \left[ \bar{R} + (N^{-2} - 5/2) \bar{g}^{\alpha\beta} \bar{q}_{\alpha\beta} \right] \bar{g}^{-1} \] (6)

or, returning to the function \( \sigma \),

\[ \bar{L} = \left[ \bar{R} - \alpha_0 \bar{g}^{\alpha\beta} \bar{\sigma}_\alpha \bar{\sigma}_\beta + e^{-(1/2)N_0} \bar{L}_m \right] \bar{g}^{-1} \] (7)

where

\[ \alpha_0 = N^2 \left( \frac{5}{2} - N^{-1} \right) > 0, \] (8)

Meanwhile,

\[ \bar{g}^{\alpha\beta} = e^{\sigma} \bar{g}^{\alpha\beta}. \] (9)

The equations of the field in a vacuum follow from (7):

\[ \bar{R}_{\alpha\beta} = \alpha_0 \bar{\sigma}_\alpha \bar{\sigma}_\beta, \] (10)

\[ \Box \sigma = 0. \] (11)

1. STATIC FIELDS

We will examine a static axially symmetric metric of the form

\[ ds^2 = e^{2\varphi} dx^2 + e^{-2\varphi} \left( d\rho^2 + dz^2 \right) + \rho^2 d\varphi^2. \] (12)

In this case, the field equations have the form [2]

\[ \Delta_0 \psi = 0, \] (13)

\[ \Delta_0 \sigma = 0, \] (14)

\[ \beta_{\rho} = \frac{1}{2} \rho \left( \psi_{\rho} \right)^2 + \psi_{\rho \rho} + \alpha_0 \left( \sigma_{\rho} \right)^2 + \alpha_0 \left( \psi_{\rho} \right)^2, \] (15)

where \( \Delta_0 \) is the "plane" Laplace operator.

We now change over from the coordinates \( \rho \) and \( z \) to the new coordinates \( x \) and \( y \):

\[ \rho = L \sqrt{(x^2 + \varepsilon)(1 - y^2)}, \] (16)

\[ z = Lxy. \] (17)

Here, \( \varepsilon = 0, \pm 1 \), and \( L \) is a constant. At \( \varepsilon = 1 \), \( x \) and \( y \) are coordinates of an oblate ellipsoid of revolution. At \( \varepsilon = 0 \), \( x \) and \( y \) are spherical coordinates, while they are coordinates of a prolate ellipsoid of revolution at \( \varepsilon = -1 \).

Equation (13) appears as follows in these coordinates: