SIMPLE MASS DISTRIBUTION
FOR THE LUNAR POTENTIAL

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Abstract. A set of twenty-one point masses gravitationally equivalent to the L1 lunar potential model is presented. By construction, the equivalence is valid only in a region of space 'sampled' by Apollo spacecraft. That region is taken to be a finite, torus-shaped shell. When used in place of the L1 model for Apollo 12 lunar orbit determination, the solution set gives spacecraft positions identical to within about 100 m.

The solution is developed in two steps: first the L1 potential is examined to determine favorable mass locations, and then the mass values are computed to force an optimum matching of the L1 potential. Therefore the solution set is ‘artificial’. It is related to the Moon’s actual mass distribution only in its similar gravitational effects in a limited region of space.

1. Introduction

A distribution of twenty-one point masses gravitationally equivalent to the L1 lunar potential model over a restricted region of space is determined in this paper. Since the results must be understood in a rather detailed context, they are deferred to Section 4. Although for practical work the distribution seems to be neither more nor less useful than the L1 model, it does offer a usable, alternative representation of the L1 model. More generally, the method employed may be used to construct a mass representation for any potential.

The object being matched, the L1 lunar potential model, is an approximation to the Moon’s gravitational potential. It has been used extensively in real-time orbit determination for recent Apollo flights. Consider a series expansion of the Moon’s potential \( V \) in terms of solid spherical harmonics. This may be written as

\[
V(r, \theta, \phi) = \left(\frac{GM}{r}\right) \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{R}{r}\right)^l P_l^m(\cos \theta) \\
\times \left[ C_{lm} \cos m\phi + S_{lm} \sin m\phi \right],
\]

in which \( GM \) is the gravitational constant times the Moon’s mass and \( R \) the mean lunar radius; \( (r, \theta, \phi) \) are the spherical polar coordinates of a field point outside the Moon, at which the potential is to be evaluated; \( P_l^m(\cos \theta) \) is the unnormalized associated Legendre function defined by Emde and Jahnke (1945); and \( C_{lm} \) and \( S_{lm} \) are the expansion coefficients of \( V \). The L1 potential is a finite approximation to the infinite series (1). It is characterized by the six non-zero coefficients listed in Table I. These coefficients were determined by a community of American selenodists after

* The values assumed for these quantities are \( GM = 4.902778 \times 10^{12} \text{ m}^3/\text{s}^2 \) and \( R = 1.73809 \times 10^6 \text{ m} \).
careful processing of doppler tracking data from Lunar Orbiter satellites (Wollenhaupt, 1970). In the region of lunar space sampled by these spacecraft, the L1 model gives a representation of the underlying lunar potential which has been satisfactory for lunar navigation and orbit determination in the Apollo program.

**TABLE I**

| Non-zero expansion coefficients characterizing the L1 lunar potential model |
|-------------------------------|-----------------|
| $C_{00}$                      | 1.0             |
| $C_{20}$                      | $-0.207108 \times 10^{-3}$ |
| $C_{22}$                      | $0.20716 \times 10^{-4}$  |
| $C_{30}$                      | $0.21 \times 10^{-4}$     |
| $C_{31}$                      | $0.34 \times 10^{-4}$     |
| $C_{33}$                      | $0.2583 \times 10^{-5}$   |

Recognizing that the Moon’s gravitational potential exists by virtue of a mass distribution, one would like to draw conclusions about the mass distribution from the present knowledge of the potential. Unfortunately, because the potential is an integral over the mass distribution, it is impossible to infer a unique distribution from knowledge of the potential alone. A classic example of this phenomenon is given by a point mass and a uniform spherical shell of equal mass, both of which have the same external potential. However, when additional information (such as extensive seismic data) about the Moon’s interior becomes available, it may be possible to use it to discriminate between various distributions determined by considering only the potential.

For the present, since there is no means for such discrimination, we will be content to determine a simple mass distribution which is gravitationally equivalent to the L1 potential model. ‘Equivalent’ is construed as requiring the L1 potential and the simple distribution to produce nearly the same state vectors at all times during lunar orbit determination for a spacecraft orbiting within a region in which the L1 potential is valid. The region is taken to be a torus-shaped shell lying within the space blocked out by the orbits studied for the construction of the L1 potential. It simulates the lunar space explored by Apollo spacecraft. ‘Simple’ is construed as requiring a smallest distribution of point masses which will produce equivalence.

The method employed involves selecting the locations of point masses within a distribution and then using a least squares integral criterion to determine the optimum mass values for the distribution. The mass values are optimum in the sense that they force a minimum of the integrated square of the difference between the L1 potential and the potential due to the selected distribution. The integration region is the torus-shaped shell just described. The mass locations may be iterated to diminish the value of the criterion integral until it is satisfactorily close to zero. Satisfactory fit is checked by applying the gravitational equivalence criterion mentioned above.