A NONLINEAR ANALYSIS OF THE MOON'S PHYSICAL
LIBRATION IN LONGITUDE

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Abstract. The Euler equations for the forced physical librations of the Moon have already been solved
by using a digital computer to perform the semi-literal mathematical manipulations. Very near
resonance, the computer solution for the physical libration in longitude is complemented by the
solution of the appropriate Duffing equation with a dissipation term. Because of its apparent proxim-
ity to a resonant frequency, the term whose argument is 2\omega – twice the mean angular distance of the
Moon's perigee from the ascending node of its orbit – is especially important. Its phase, which soon
should be measurable, is related to the Moon's anelasticity. The term's frequency, in units of the
sidereal month, increases as the semi-major axis of the Moon's orbit about the Earth increases. Using
the Moon's mechanical ellipticity of Koziel and the rate of increase of the semi-major axis of MacDo-
nald, it is estimated that the 2\omega term will cross the resonant frequency in 130 million years and, if the
rate of energy dissipation is sufficiently low, a transient libration will be induced.

1. Introduction

The Euler equations for the forced physical librations of a perfectly elastic Moon have
been solved by using a digital computer to perform the semi-literal mathematical manipu-
lations (Eckhardt, 1970). Very near a three year resonance, the computer solution
for the physical libration in longitude is inadequate because it neglects nonlinear and
dissipation effects. To complement the computer solution, we have investigated the
physical libration in longitude near this resonance, taking into account nonlinear and
dissipation terms. We have considered also the secular increase in the period of the
relevant torque term and its effect, as the period passes through resonance, on the
libration. Our method of investigation has been analytic, but since our goal has been
to achieve a qualitative insight into libration near resonance, we have ignored many
small effects and we have made some broad extrapolations. If our theory were more
complete, our results would be more exact, but we believe that our understanding of
the problem would not be significantly improved.

2. The Euler Equation for Libration in Longitude

The Euler dynamical equation for the physical libration in longitude, \( \tau \), may be
approximated by

\[
\frac{1}{n^2} \frac{d^2 \tau}{dt^2} = 2.977 \gamma \left[ \frac{1}{2} \sin 2(s - \tau) + u \right],
\]

where \( n \) is the mean motion of the Moon; \( \gamma \) has its usual connotation as the relevant moment of inertia ratio (approximately \( 2.3 \times 10^{-4} \)); \( s = s(t) \) is the center equation and inequalities in the lunar longitude; and \( u = u(t) \) is composed of cross terms involving the center equation and inequalities in the lunar longitude and lunar parallax (Eckhardt, 1970).

According to lunar theory \( s \) is, at most, approximately 0.1 radians; let us suppose that \( \tau \) is small enough that the following approximation is valid,

\[
\frac{1}{2} \sin^2(s - \dot{\theta}) \approx \frac{1}{2} \left( 1 - 2\sin^2s \right) \cos 2\dot{\theta} - \frac{1}{2} \cos^2s \sin 2\dot{\theta}
\]

\[
= \frac{1}{2} \left( 1 - 2\tau^2 \right) \sin 2s - \cos 2s (\tau - \frac{3}{2}\tau^3),
\]

where \( \cos 2s = 1 - 2\sin^2s = 0.985 \) is the mean value of \( \cos 2s \). It is known by observation that the amplitude of \( \dot{\theta} \) is now less than \( 10^{-3} \) radians, so there is little loss in precision in the linearization \( \frac{1}{2} \sin 2(s - \tau) \approx \frac{1}{2} \sin 2s - 0.985\tau \) which, inserted into (1), gives the linear differential equation

\[
\frac{1}{n^2} \frac{d^2 \tau}{dt^2} + 2.977 \gamma \tau = 2.977 \frac{1}{2} \sin 2s + u. \tag{3}
\]

The term in brackets on the RHS of (3) can be expanded into a Fourier sine series, \( \Sigma \xi_i \sin \xi_i \), whose arguments are elements of the additive group generated by the Hansen or Delaunay arguments. The particular solution to (3) is then given by

\[
\tau = \Sigma \xi_i \xi_i = \Sigma \frac{h_i H_i}{h_i - 1} \sin \xi_i,
\]

where

\[
h_i = 2.977 \\left[ \frac{1}{n} \frac{d \xi_i}{dt} \right]^2.
\]

For the coefficient of the \( \tau \) term whose argument is \( 2\omega = 2F - 2l \) (twice the mean distance of the Moon’s perigee from the ascending node of the Moon’s orbit) there is a singularity at \( \gamma = \gamma \approx 2.126 \times 10^{-4} \). Because \( \gamma \) is actually very close to \( \gamma \), the coefficient of this three year libration term is considerably amplified over the one arc-second amplitude of the corresponding forcing term, and the sign of the coefficient depends on the sign of \( \gamma - \gamma \); that is, the sign is ambiguous. We shall examine this term by dropping the subscript \( i \), letting \( \xi = 2\omega \), and considering the equation

\[
\frac{d^2 \tau}{d\xi^2} + h \tau = HH \sin \xi.
\]