APPLICATION OF APPROXIMATE SYMMETRIES TO
THE CONSTRUCTION OF SOLUTIONS OF CLASSICAL
AND QUANTUM HAMILTONIAN SYSTEMS

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A method for integration of classical and quantum Hamiltonian systems, based on the use of approximate symmetries, is proposed in this paper. The proposed method is similar to the averaging method in classical mechanics; however, it does not use canonical transformations to the variables "action-angle." This allows one in some cases to apply this method to quantum equations. A nontrivial example is analyzed in the paper.

An approximate analysis of equations with a small parameter is performed using expansion methods with respect to the parameter. A review of these methods can be found in [1]. Many of these methods originally appeared as special tricks for solving concrete problems, and only later, they were generalized and put on a rigorous foundation. At the same time, certain methods, especially those applied to nonlinear equations, have not been put on a rigorous basis yet. Successful use of these methods is possible on the basis of additional information, which reflects nontrivial features of the system under investigation.

In order to develop approximate methods, it seems promising to use the ideas from the symmetry analysis because symmetry properties reflect the most significant features of the system under investigation. We also emphasize that these properties serve as a basis for construction of a method for exact integration of equations. One of the first systematic studies in this direction is the work of N. Kh. Ibragimov and coworker, in which approximate symmetries with a small parameter have been studied (see review [2]).

In this paper, approximate symmetries are used to study Hamiltonian systems, which are close to integrable ones (when the integrated Hamiltonian is perturbed by a small correction, which violates the integrable character), in the context of correspondence between classical and quantum systems. This theme has various nontrivial aspects, which are being actively studied at the present time. We note, for example, stochastic manifestation of classical dynamic systems in their quantum properties [3].

For classical Hamiltonian systems, there exist integration methods based on special canonical transformations [4]. A direct application of these methods to quantum theory encounters certain difficulties, which can sometimes be avoided with the help of approximate symmetries in conjunction with a certain averaging procedure.

In a classical Hamiltonian system in the canonical variables $(I, \psi)$ of the type action-angle, which are constructed on the basis of the unperturbed Hamiltonian, averaging over the angle variable $\psi$ is used when constructing new canonical variables $(J, \psi)$. Within variables $(J, \psi)$, the Hamiltonian consists of the averaged part, which depends on $J$ and which contains small corrections, and the perturbing part, which depends on $(J, \psi)$. Moreover, the latter is of a higher order than the analogous initial part. From the point of view of the symmetry theory, variables $I$ are exact integrals of the unperturbed Hamiltonian, and variables $J$ are approximate integrals for the perturbed Hamiltonian and exact integrals for the averaged Hamiltonian. Variables $I$ determine the Abel Hamiltonian symmetry group [5] of the Hamiltonian system for the unperturbed Hamiltonian. Variables $J$ give a similar group of approximate symmetries for the averaged Hamiltonian.

In this paper, a method for approximate integration of Hamiltonian systems is proposed, in which canonical variables are found with the help of the notion of approximate symmetries; this allow one to make a transformation from a classical to a corresponding quantum system.

In order to obtain a solution to the quantum equations in an explicit form, one searches for approximate symmetries in the class of differential operators thereby limiting the form of the perturbation potential.

Let us consider for simplicity a two dimensional classical Hamiltonian system with the Hamiltonian $H = H_0 + \epsilon V$, where the unperturbed Hamiltonian $H_0$ allows for the motion integral $X$, i.e. $\{H_0, X\} = 0$. Here and below, $\{,\}$ is the Poisson
bracket, and the motion is assumed finite, i.e. the trajectory of the unperturbed system are situated on the two dimensional Liouville torus, and this torus does not destroy the perturbation. In this paper, we consider the first correction of the perturbation theory; generalization to higher order perturbations does not cause problems.

We shall search for the approximate motion integral $Z$ in the form $Z = X + eY$. Then,

$$\{H, Z\} = e \{V, X\} + \{H_o, Y\} + e^2 \{V, Y\}.$$  

The equation for $Y$ is

$$\{H_o, Y\} = \{X, V\}.$$  

Because integral $X$ is known for the Hamiltonian $H_o$ Eq. (1), which is a linear inhomogeneous equation with respect to function $Y$, can be resolved in quadrature. A similar situation takes place in the multidimensional case. Let us denote $f(x, p) = \{V, Y\}$, then $\{H, Z\} = e^2f$. We require that $f$ be bound and

$$\langle f \rangle \equiv \lim_{\tau \to \infty} \frac{1}{T} \int_0^T f(x(t), p(t)) \, dt = 0.$$

Here, $x(t)$ and $p(t)$ determine the law of motion of the unperturbed Hamiltonian system. Satisfying these requirements means that, although $Z$ is not an exact motion integral of the perturbed system, the true trajectories do not deviate far from the surface $Z(x, p) = \text{const.}$

Let us replace variables in the phase space $x = x(\alpha, Z), p = p(\alpha, Z)$ so that the Hamiltonian vector field $\text{grad}(Z) = \frac{\partial Z}{\partial p}(\frac{\partial}{\partial x}) - \frac{\partial Z}{\partial x}(\frac{\partial}{\partial p})$, in the new variables takes the form $\text{grad}(Z) = \frac{\partial}{\partial \alpha}$. Variables $Z$ and $\alpha$ are canonically conjugate; moreover, because $Z$ is a slow variable, then $\alpha$ is a fast variable. Let us average the perturbed Hamiltonian over the fast variable $\alpha$:

$$\langle H \rangle \equiv \frac{1}{T} \int_0^T H \, d\alpha.$$  

Here, $T$ is the period of function $H$ with respect to variable $\alpha$. If $H$ is nonperiodic, then in Eq. (2) one should take the limit $T \to \infty$. Then $\{Z, \langle H \rangle\} = \text{grad}(Z)(\langle H \rangle) = 0$, i.e. the averaged Hamiltonian $\langle H \rangle$ allows the exact motion integral $Z$. This procedure corresponds to averaging of the Hamiltonian in the KAM (Kolmogorov—Arnol’d—Moser) theory [4], without explicit substitution of canonical transformations. Taking into account symmetrical nature of the variable $Z$, we can establish a correspondence between this variable and the symmetry operator in quantum mechanics, and apply the described procedure to quantum mechanical equations.

We shall illustrate this approach using an example. Let us consider an unperturbed Hamiltonian $H_0$, which describes two dimensional motion of a material point having unit mass in a radially symmetric field having the potential $u(r)$ in cylindrical coordinates $(r, \phi)$:

$$H_0 = \frac{1}{2} p_r^2 + \frac{1}{2r^2} p_{\phi}^2 + u(r).$$

$H_0$ allows the motion integral $X = p_r$. We obtain from Eq. (1)

$$V = \cos \phi, \quad V = -\sin \phi u'(r), \quad u'(r) = du(r)/dr.$$  

We note that the condition for applicability of perturbation theory has the form $\epsilon/r_{\text{min}} < 1$. Let us introduce new coordinates

$$z = \arcsin \frac{r \sin \phi - \epsilon}{\sqrt{1 + r^2 - 2r \sin \phi}}, \quad z \in [0, 2\pi),$$  

$$\rho = \sqrt{r(r - 2\epsilon \sin \phi)}, \quad \rho \in [0, \infty).$$  

(3)