Analytical Solution for Impedance Change Due to Flaws in Eddy Current Testing

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Green’s function is used in order to derive the analytical solution for the change in impedance due to a presence of the flaws in a conductor. This solution is applied to a cylindrical flaw and a spherical flaw whose radii are much smaller than the radius of the test coil. For both cases, the change in impedance is obtained within Born’s limit.

KEY WORDS: Eddy current; impedance; flaw; Green’s function; Born’s limit; NDE.

1. INTRODUCTION

We present a theoretical study for the change in impedance due to the presence of flaws in a conductor which is linear, isotropic, and homogeneous. Through the years, this problem has been studied by many people. They used $\Delta Z$ formula to obtain analytical solutions for impedance changes.

In this paper, we formulate the problem using Green’s function instead of $\Delta Z$ formula. Green’s function method is not only a simple method to derive solutions for the impedance change but also gives much information about flaws. In applying this formula to a conductor containing a flaw, Born approximation is used. The main attraction of the Born approximation is its simplicity. In Section 2, Green’s function for a semi-infinite conductor is presented, and in Section 3, the result is employed in order to treat flaws within the Born’s limit.

2. GREEN’S FUNCTION

The differential equation in a cylindrical coordinate for function $G$ in an isotropic, linear and homogeneous medium due to delta-function current density $\delta(r-r')\delta(z+z')$ is

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} + \frac{\partial^2 G}{\partial z^2} - \frac{G}{r^2} - j\omega \mu \sigma G = 0 \tag{1}$$

In Eq. (1), axial symmetry is assumed as shown in Fig. 1. In fact, function $G$ represents the vector potential due to delta-function current density and is used as Green’s function for the calculation of the changes of vector potential due to flaws. This problem can be considered with three different regions of the same permeability.

The differential equation in air (region I) is

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} + \frac{\partial^2 G}{\partial z^2} - \frac{G}{r^2} = 0 \tag{2}$$

since there is no current source, and the conductivity is zero. The differential equation in a conductor (regions II and III) is

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} + \frac{\partial^2 G}{\partial z^2} - \frac{G}{r^2} - j\omega \mu \sigma G = 0 \tag{3}$$

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since the delta-function current source lies in the interface between region II and region III. The solutions in each region become
\[ G_I(r, z) = \int_0^{\infty} B_1(\alpha)e^{-\alpha z}J_1(\alpha r) \, d\alpha \] (4)
\[ G_{II}(r, z) = \int_0^{\infty} [C_2(\alpha)e^{\alpha z} + B_2(\alpha)e^{-\alpha z}] \times J_1(\alpha r) \, d\alpha \] (5)
\[ G_{III}(r, z) = \int_0^{\infty} C_3(\alpha)e^{\alpha z}J_1(\alpha r) \, d\alpha \] (6)
where \( \alpha^2 = \alpha^2 + j\omega \mu \sigma \).

The coefficients in Eqs. (4)-(6) can be determined from the following boundary conditions:
1. \( G_I(r, 0) = G_{II}(r, 0) \) (7)
2. \( \frac{\partial G_I}{\partial z}(r, 0) = \frac{\partial G_{II}}{\partial z}(r, 0) \) (8)
3. \( G_{II}(r, -z') = G_{III}(r, -z') \) (9)
4. \( \frac{\partial G_{II}}{\partial z}(r, -z') + \delta(r - r') = \frac{\partial G_{III}}{\partial z}(r, -z') \) (10)

For the present purpose we only need Green's function in region I, \( G_I \). After a simple calculation \( G_I \) becomes
\[ G_I(r, r', z, -z') = \int_0^{\infty} \frac{\alpha}{\alpha_i + \alpha} r'J_1(\alpha r')e^{-\alpha z'}e^{-\alpha z}J_1(\alpha r) \, d\alpha \] (11)

3. TREATMENT OF FLAWS

Consider a conductor which contains a small flaw as shown in Fig. 2. The differential equation for the vector potential \( A \) is
\[ \frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - A - j\omega \mu \sigma A = 0 \] (12)
We can write the conductivity of a flaw, \( \sigma' \), as
\[ \sigma' = \sigma + \delta \sigma \theta(r) \] (13)
thus defining the perturbation
\[ \delta \sigma = \sigma' - \sigma \] (14)
with \( \theta(r) \) satisfying
\[ \theta(r) = 1 \text{ for } r \text{ in a flaw} \]
\[ = 0 \text{ for } r \text{ not in a flaw} \] (15)

Then Eq. (12) becomes
\[ \frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - A - j\omega \mu \sigma A - j\delta \sigma \omega \mu A(r) = 0 \] (16)
The solution of Eq. (16) for the vector potential \( A \) is evaluated from Green's convolution as follows:
\[ A_I = A_{I0} - j\delta \sigma \omega \mu \int G(r, r', z, z') \times A_{III}(r', z') \theta(r') \, dr' \, dz' \] (17)
Since from Eq. (17) we can not directly obtain information on flaws we necessarily seek approximate methods; obviously these entail the development of an approximation for \( A_{III} \) inside the flaw. To obtain the approximation, we expand Eq. (17) in Born series...