EFFECTS OF THE ORBITAL ECCENTRICITY ON THE
EQUIPOTENTIAL SURFACES IN BINARY SYSTEMS

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Abstract. In the frame of the elliptical restricted three-body problem, the differential equations for the motion of an infinitesimal body are established. In spite of the lack of Jacobi's integral, for fixed values of the true anomaly (v = 0°, 90°, and 180°), particular results were obtained. The pulsational character of the equipotential surfaces is evident.

1. Introduction

As was mentioned by Kopal and Lyttleton (1963), if the Keplerian orbit of the two finite masses $m_1$ and $m_2$ is eccentric, the geometrical properties of the equipotential surfaces are bound to depend on the time. In such a case '... in order to establish the instantaneous shape of the zero-velocity surface, we would need to know the value of $d\Omega$ at each value of $t$ (or $v$) all along any selected path' (Kopal, 1989, p. 35).

Such a subject is studied in the frame of the Elliptical Restricted Three-Body Problem, and the aim of the present note is to resume the effects of the orbital eccentricity on the equipotential surfaces. Here the two finite bodies will be considered as mass-points.

2. The Elliptical Three-Body Problem in an Inertial Coordinate System

Let us choose an inertial barycentric rectangular coordinate system $(X', Y', Z')$, where the $X'$-axis is directed towards the periastron, while $X'Y'$-plane coincides with orbital plane. In such a case, the differential equations for the motion of the infinitesimal body may be written in the form

$$\frac{d^2X'}{dt^2} = -G \frac{m_1}{r_1^3} (X' + R_1 \cos v) - G \frac{m_2}{r_2^3} (X' - R_2 \cos v),$$

$$\frac{d^2Y'}{dt^2} = -G \frac{m_1}{r_1^3} \ Y' - G \frac{m_2}{r_2^3} \ Y', \quad (1)$$

$$\frac{d^2Z'}{dt^2} = -G \frac{m_1}{r_1^3} \ Z' - G \frac{m_2}{r_2^3} \ Z';$$

with

$$r_1^2 = (X' + R_1 \cos v)^2 + (Y')^2 + (Z')^2,$$

$$r_2^2 = (X' - R_2 \cos v)^2 + (Y')^2 + (Z')^2 \ ; \quad (2)$$
where the positions of the two finite bodies are:

\[ S_1(-R_1 \cos v, -R_1 \sin v), \quad S_2(R_2 \cos v, R_2 \sin v), \]

where \( v \) is the true anomaly; \( e \), orbital eccentricity; \( A \), semi-major axis,

\[
R_1 = \frac{m_2}{m_1 + m_2} \frac{A(1 - e^2)}{1 + e \cos v}, \quad R_2 = \frac{m_1}{m_1 + m_2} \frac{A(1 - e^2)}{1 + e \cos v},
\]

\[
R = R_1 + R_2 = \frac{A(1 - e^2)}{1 + e \cos v}.
\]

3. The Use of a Non-uniform Rotating Coordinate System

Now the barycentric coordinate system \((X, Y, Z)\) is considered in rotation with the binary system. Here the true anomaly \( v \) will be an 'angular' function, reckoned from the periastron passage. In such conditions the transformation equations between \((X', Y', Z')\) and \((X, Y, Z)\) are in the form

\[
\begin{align*}
X' &= X \cos v - Y \sin v, \\
Y' &= X \sin v + Y \cos v, \\
Z' &= Z;
\end{align*}
\]

and Equations (1) become

\[
\begin{align*}
\frac{d^2X}{dt^2} - 2 \left( \frac{d^2v}{dt^2} \right) \frac{dY}{dt} &= \left( \frac{d^2v}{dt^2} \right)^2 X - G \frac{m_1}{r_1^3} (X + R_1) - \\
&- G \frac{m_2}{r_2^3} (X - R_2) + Y \frac{d^2v}{dt^2}, \\
\frac{d^2Y}{dt^2} + 2 \left( \frac{d^2v}{dt^2} \right) \frac{dX}{dt} &= \left( \frac{d^2v}{dt^2} \right)^2 Y - G \frac{m_1}{r_1^3} Y - G \frac{m_2}{r_2^3} Y - X \frac{d^2v}{dt^2}, \\
\frac{d^2Z}{dt^2} &= - G \frac{m_1}{r_1^3} Z - G \frac{m_2}{r_2^3} Z;
\end{align*}
\]

with

\[
r_1^2 = (X + R_1)^2 + Y^2 + Z^2, \quad r_2^2 = (X - R_2)^2 + Y^2 + Z^2.
\]

In the first two Equations (5) the second terms on the left-hand side represent Coriolis accelerations. The first terms on the right-hand side are centrifugal effects. The next two terms are the gravitational effects, while the fourth terms represent the acceleration normal to the radius vector due to the non-uniform rotation of the coordinate system \((X, Y, Z)\).