1. The basic aim of the study of the mechanical behavior of blood vessels is obtaining definite equations (or a protein function, from which these equations follow directly) relating terms describing the stress-deformation state of the blood vessel. This information is required for solving any concrete boundary-value problem in circulation mechanics and is also important for the design and use of artificial blood vessels or in the mutual replacement of various segments of the circulatory system.

Many studies on the rheology of blood vessels have shown that there is sufficient basis to consider arterial tissue as a nonlinear, elastic, incompressible material. It has curvilinear orthotropy relative to the axes in the axial, circumferential, and radial directions of an arterial segment considered as a cylindrical shell. Hence, under conditions of axially symmetrical physiological loading (transmural pressure and the axial force produced by the action of the surrounding tissues), the artery is in an axially symmetrical stress-deformation state. In actuality, the absence of shear deformations when a segment of the vessel is subjected to axial stretching and inflation by internal pressure is used as experimental support for the existence of the curvilinear orthotropy mentioned above [1].

In most cases, we must only have the relationship between the normal stresses along the orthotropy axes (major stresses) and the corresponding tensile deformations in order to describe the mechanical behavior of blood vessels. Considering the general geometry and physical nonlinearity of the tissue of the vessel wall and assuming the material is elastic, we must only know the function for the specific energy of deformation (elastic potential) in order to obtain the specific equations. In the axially symmetrical case discussed above, it is the function of the components of the deformation tensor in the axial, circumferential, and radial directions which are mutually independent as a consequence of the incompressibility of the material. The appearance of this function and the values of the constants within it are determined experimentally for the specific arterial vessel (see previous work [2-5]). However, conditions exist, under which knowledge of the dependence between the major stresses and the corresponding deformation along the orthotropy axes is not sufficient for describing the mechanical behavior of the medium. This occurs in those cases in which the application of stress on a vessel is not axially symmetrical. Such situations arise in segments of the aorta near the heart as a consequence of the nonsymmetrical blood flow from the cardiac muscle and, most of all, this is pronounced under conditions in which nonsymmetrical external stresses act on the blood vessel. In order to solve such boundary-value problems and also for designing artificial substitutes for blood vessels, specific equations are required in the general case. To find such solutions, we must initially solve the boundary-value problem representing experiments.

Fig. 1. Coordinate scheme for a blood vessel in the undeformed state $B_0$ and deformed state $B$. 


tally realized stress conditions. In the present article, we consider the problem of simultaneous stress on a blood vessel by axial tensile force, internal pressure, and a torsional moment.

2. The arterial segment of a vessel is considered as a round cylindrical shell. The material is assumed elastic, incompressible, and curvilinearly orthotropic. The action of internal pressure, axial force, and torque applied at the ends causes a finite uniform deformation of the vessel. In a certain stress–deformation state, the vessel is considered a membrane and bending rigidity is ignored due to large deformation in the vessel wall.

Let the membrane have length $L_0$, radius $R_0$, and thickness $h_0$ (Fig. 1) in the natural undeformed state $B_0$ when stress is not applied to the membrane. The position of an arbitrary point $N_0$ in the middle surface is conveniently defined using a curvilinear coordinate system $\theta^\alpha (\alpha = 1, 2)$, which is shown in Fig. 1. The coordinate $\theta^1$ is the distance from the plane $0xy$ measured along axis $z$ and $\theta^2$ is the distance from plane $0xz$ along the circumference. The basis vectors $a_\alpha$ in this system coincide with the orthotropy directions and the corresponding metric surface tensor has the components

$$ a_{\alpha \beta} = \delta_{\alpha \beta}; \quad \delta_{\alpha \beta} = \begin{cases} 1, & \alpha = \beta; \\
0, & \alpha \neq \beta. \end{cases} $$

In the deformed state $B$, when internal pressure $p$, axial force $F$, and torque $M$ are effective, the membrane has length $L = \lambda_1 L_0$, the radius of the mean surface $R = \lambda_2 R_0$, and the cross section perpendicular to the cylinder axis is twisted in its plane by an angle proportional to the distance from one end. The relative torsional angle is termed $\psi$ and the elongations in the axial and circumferential directions are termed $\lambda_1$ and $\lambda_2$. The position of point $N$ of the middle surface in state $B$ may be described using the curvilinear coordinate system $\theta^\alpha$ whose basis vectors are unitary and mutually perpendicular such that the metric surface tensor has components $g_{\alpha \beta}$. The proposed final deformation is given using the relationship between the coordinates of the same point in the two states in the following form:

$$ \theta^1 = \lambda_1 \theta^1; \quad \theta^2 = \lambda_2 \theta^2 + \psi \lambda_1 \theta^1. $$

Let us give the components of the metric surface tensor in state $B$ relative to the $\theta^\alpha$ coordinate system selected as the convective by $A_{\alpha \beta}$. Using the equation for transformation of the covariant tensor $A_{\alpha \beta}$, considering Eq. (1), we obtain

$$ A_{\alpha \beta} = \begin{bmatrix} \lambda_1^2 (1 + \psi^3) & \psi \lambda_1 \lambda_2 \\ \psi \lambda_1 \lambda_2 & \lambda_2^2 \end{bmatrix}. $$

According to the theory of finite transformations [6], the deformation tensor characterizing the deformation state of the middle surface of the membrane has the following components:

$$ e_{\alpha \beta} = \frac{1}{2} (A_{\alpha \beta} - a_{\alpha \beta}) = \begin{bmatrix} \frac{1}{2} [\lambda_1^2 (1 + \psi^3) - 1] & \frac{\psi \lambda_1 \lambda_2}{2} \\ \frac{\psi \lambda_1 \lambda_2}{2} & \frac{1}{2} (\lambda_2^2 - 1) \end{bmatrix}. $$

Transition of the membrane from state $B_0$ to state $B$ is also characterized by change in the thickness $h = \lambda_3 h_0$, where $\lambda_3$ is the elongation relative to thickness. If we introduce a radial component $\theta^3$ which is measured at any point relative to the normal to the middle surface in the undeformed state, then the corresponding deformations are: $e_{33} = \frac{1}{2} (\lambda_3^2 - 1)$, $e_{13} = e_{23} = 0$. From the condition of incompressibility of the material $\lambda_3 \times \det A_{\alpha \beta} / \det a_{\alpha \beta} = 1$, in this case we obtain that $\lambda_3 = 1/(\lambda_1 \lambda_2)$ and, thus, the components $e_{\alpha \beta}$ and $e_{33}$ are independent.

The stressed state of the membrane is determined by the stress tensor $\tau^{\alpha \beta}$ which acts in the cross section normal to the surface and is taken constant over the shell thickness. According to the theory of elastic membranes $\tau^{33} \ll \tau^{\alpha \beta}$, and $\tau^{\alpha 3} = 0$. Thus, it is convenient to introduce a membrane force tensor relative to a unit length of the deformed middle surface, $n^{\alpha \beta} = h \tau^{\alpha \beta}$.

The mechanical behavior of a nonlinear membrane is completely determined when the elastic potential $W$ is given as a function of dimensionless components of the deformations relative to the coordinate system whose axes coincide with the direction of curvilinear orthotropy in the undeformed object. In the selected convective coordinate system $\theta^\alpha$, $W = W(e_{11}, e_{22}, e_{33}, e_{12})$ for an elastic membrane from the orthotropy of the material according to Green and Adkins [7]. It may be shown (see Appendix) that for an incompressible membrane knowledge of the elastic potential as a function of the deformation of the middle surface is sufficient, i.e., $\tilde{W} = W(e_{11}, e_{22}, e_{12})$. Then

$$ \tau^{\alpha \beta} = \frac{\partial \tilde{W}}{\partial e_{\alpha \beta}}; \quad \tau^{12} = e_{12} \frac{\partial \tilde{W}}{\partial (e_{12})}. $$

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