Leaky modes in optical waveguides

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Solutions of the scalar wave equation are presented giving, explicitly, the excitation and subsequent attenuation of leaky modes in optical waveguides.

1. Introduction
The study of leaky modes in optical waveguides has been the subject of much interest during the last decade (see e.g. [1–3]; there have been a very large number of papers on leaky modes, but most of the important results have been very well described in [1, 2]). Not only are leaky modes technologically important for our understanding of the losses in waveguides, but they are also very interesting from the mathematical point of view. Leaky modes are usually referred to as discrete solutions of the eigenvalue equation below cutoff with the corresponding propagation constant being complex so that the mode suffers attenuation as it propagates along the waveguide (see e.g. [1] p. 320). The complete expression for the field in an optical waveguide consists of a sum of bound-mode terms plus an integer over the continuous spectrum or radiation modes. Complex variable techniques may be used to manipulate the integral into a discrete sum of pole terms. This process is carried out for the optical waveguide in Chapter 26 of [2] where it is shown that the discrete terms are in fact the leaky-mode contributions. In this paper we use the same approach, but execute the mathematics in a more explicit and less formal manner so that the reader can easily appreciate the origin of leaky modes.

We first consider the radiation modes of the waveguide and show that certain radiation modes, the ‘quasi-modes’, have only an evanescently decaying connection into the oscillating field in the cladding. We show that if the waveguide is excited by a field that corresponds with the field of a quasi-mode in the core region, but is almost zero elsewhere, then power is retained in the core, and we may think of a field propagating in the manner of a bound mode except that an attenuation coefficient is required. Thus, we arrive at the concept of a ‘leaky mode’. We derive an expression for the attenuation coefficient which is of a form similar to the one previously obtained (see e.g. [2–5]).

In Section 2.1, we briefly formulate the waveguide problem. In Section 2.2, we consider the exact solution for a specific profile which will rigorously correspond to a slab waveguide with a W-type profile. This analysis brings out most of the salient aspects of the leaky modes. In Section 2.3, we present the WKB analysis which is valid (and has been extensively used) in the study of multimode optical fibres characterized by smooth refractive-index profiles (see e.g. [1]). The mathematical manipulations are similar to those used in the tunnelling problem in quantum mechanics (see e.g. [6, 7]). We should point out that the use of the WKB formalism means that our results will be related to the full asymptotic theory of geometric optics and leaky rays [4, 5]. Further, we will deal only with a scalar version of waveguide theory.

2. Theory
2.1. The waveguide problem
In the weakly guiding approximation, the transverse component of the electric (or magnetic) field satisfies the scalar wave equation (see e.g. [8])

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\[ \nabla^2 \psi + k_0^2 n^2(r) \psi = 0 \]  
\( (1) \)

where \( k_0 \) represents the free-space wave number and \( n(r) \) represents the refractive-index profile; we will be using the cylindrical system of co-ordinates \( (r, \phi, z) \). Writing the field in terms of separated variables

\[ \psi(r, \phi, z) = F(r) \left[ \frac{1}{(2\pi)^{1/2}} \exp(i\phi) \right] \exp(i\beta z) \]  
\( (2) \)

we obtain the radial equation

\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dF}{dr} \right) + \left[ k_0^2 n^2(r) - \beta^2 - \frac{l^2}{r^2} \right] F(r) = 0 \]  
\( (3) \)

with \( l = 0 \pm 1, \pm 2, \ldots \). We define a new variable \( G(r) \) through the equation

\[ G(r) = r^{l/2} F(r) \]  
\( (4) \)

so that Equation 3 takes the form of a one-dimensional equation

\[ \frac{d^2 G}{dr^2} + \left[ k_0^2 n^2(r) - \beta^2 - \frac{l^2 - \frac{1}{4}}{r^2} \right] G(r) = 0 \]  
\( (5) \)

Obviously, for the field to be well-behaved at \( r = 0 \) we must have

\[ G(0) = 0 \]  
\( (6) \)

For the sake of simplicity we consider a step-index fibre with the following refractive-index distribution

\[ n = \begin{cases} n_1 & 0 < r < a \\ n_2 & r > a \end{cases} \]  
\( (7) \)

The analysis can be easily extended to graded-index fibres. Using Equation 7 we write Equation 5 in the form

\[ \frac{d^2 G}{dr^2} + [\varepsilon - f(r)] G(r) = 0 \]  
\( (8) \)

where

\[ \varepsilon = \frac{k_0^2 n_1^2 - \beta^2}{U^2/a^2} \]  
\[ U^2 = (k_0^2 n_1^2 - \beta^2)a^2 \]  
\( (9) \)

and

\[ f(r) = \begin{cases} \frac{l^2 - \frac{1}{4}}{r^2} & 0 < r < a \\ \frac{l^2 - \frac{1}{4}}{r^2} + k_0^2 (n_1^2 - n_2^2) & r > a \end{cases} \]  
\( (10) \)

We consider \( l > 1 \) so that as \( r \to 0, f(r) \to \infty \) (see Fig. 1a). On the other hand, as \( r \to \infty, f(r) \) tends to the constant value.

\[ f_{\infty} = k_0^2 (n_1^2 - n_2^2) \]  
\( (11) \)

At \( r = a, f(r) \) has a discontinuity as shown in Fig. 1a with

\[ f_1 = \frac{l^2 - \frac{1}{4}}{a^2} \quad \text{and} \quad f_2 = \frac{l^2 - \frac{1}{4}}{a^2^2} + k_0^2 (n_1^2 - n_2^2) \]  
\( (12) \)

312