Analysis of an Upscaling Method Based on Conservation of Dissipation

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(Received: 4 October 1993; in final form: 17 June 1994)

Abstract. In this paper, a scaling method based on conservation of dissipation and use of periodic boundary conditions is presented. We prove that the method leads to a symmetric positive definite tensor. We also show that the method is identical to the method of Dudovsky and Chung, therefore an important property of the latter method is proved. Some other existing methods are also discussed in terms of their boundary conditions. For this purpose, the concepts of basis and class of boundary conditions for scaling methods are introduced.

Key words: Upscaling, full tensor, boundary conditions.

1. Introduction

Scaling methods based on fine-scale flux simulation have been suggested by several authors [4, 7, 10, 19]. These methods seek an upscaled (representative, effective) permeability tensor \( K = \{ K_{ij} \}, i, j = 1, 2, 3 \), for a certain volume by numerical simulation of flow through that volume. To be more concrete, let \( k_b, b = 1, 2, \ldots N \), represent a set of fine-grid permeability tensors forming a rectangular box \( \Omega \) in \( R^3 \) with boundary \( \Gamma \). Darcy's law for one-phase gravity-free, incompressible flow in a porous medium leads to the following boundary value problem for the pressure \( p \) [16]:

\[
\nabla \cdot (k(x) \nabla p) = 0, \quad \text{in} \ \Omega,
\]

\[
p \quad \text{and/or} \quad u \cdot n \quad \text{specified on} \ \Gamma,
\]

where \( k(x) \) varies in space \( x = (x_1, x_2, x_3) \). The Darcy velocity is given by \( u = -k(x) \nabla p \), and \( n \) is the unit normal vector pointing outwards from \( \Omega \). The viscosity is a constant factor in the equation and will not affect the upscaled permeability. It is therefore taken to be unity.

For the methods considered here, the upscaled permeability tensor \( K \) is the one that reproduce (or approximate) the total fluxes across each of the surfaces of the block.

The outline of this paper is as follows: In Section 2 we discuss and analyze an upscaling technique based on conservation of dissipation [18]. The technique is based on a weak form of Darcy's law. The scaling criterion is based on conservation of dissipation between the fine and the coarse scale. We show that the use of periodic
boundary conditions leads to a method identical to the method of Durlofsky and Chung [4, 5]. Using the weak form, we are able to prove that the method leads to a positive definite symmetric tensor. We also introduce the concepts of basis and class of Dirichlet–Neumann problems for upscaling methods. These concepts are used for discussing some existing techniques and their assumptions which is the theme of Section 3.

2. Scaling Technique Based on Conservation of Dissipation

Darcy's law on the fine and coarse scales is written

\[ u = -k \nabla p, \]
\[ U = -K \nabla P. \]

The upscaling technique is based on a weak form of Darcy's law. The permeability tensor at the coarse scale is assumed diagonal and positive definite. The upscaled permeability tensor is assumed invertible. We multiply (2) by \((k^{-1}u)^t\) where the superscript \(t\) denotes the transpose, and use partial integration:

\[ \int_{\Omega} (k^{-1}u)^t u \, d\Omega = - \int_{\Gamma} p u \cdot n \, d\Gamma + \int_{\Omega} p \nabla \cdot u \, d\Omega. \]

The last integral in (4) vanishes as a consequence of the incompressibility constraint \(\nabla \cdot u = 0\). Using Darcy's law (2) we may write (4) as

\[ \int_{\Omega} \nabla (p^t k \nabla p) \, d\Omega = - \int_{\Gamma} p U \cdot n \, d\Gamma. \]

An identical derivation for (3) leads to a corresponding weak form

\[ \int_{\Omega} \nabla (p^t K \nabla P) \, d\Omega = - \int_{\Gamma} P U \cdot n \, d\Gamma. \]

Consider now the following upscaling criterion for each simulator block \(\Omega\):

\[ \text{Find } K = \text{constant for each block } \Omega \text{ such that} \]

\[ \int_{\Omega} \nabla (p^t K \nabla P) \, d\Omega = \int_{\Omega} \nabla (p^t k \nabla p) \, d\Omega. \]

\(\nabla (p^t k \nabla p)\) is the dissipation density and represent loss of mechanical energy pr. unit volume and time. The upscaling criterion given by (7) has the following physical explanation: The work pr. unit time needed in order to force the fluid through the block \(\Omega\), is equal to the dissipation of energy caused by the friction between the fluid and solid rock. The criterion conserves the dissipation energy between the fine and the coarse grids.