VARIEDIES OF NEW CLASSES OF INTERIOR SOLUTIONS IN
GENERAL RELATIVITY

D.N. PANT

Department of Mathematics, Kumaun University,
Nainital, India

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Abstract. In this paper we present a method of obtaining varieties of new classes of exact solutions representing static balls of perfect fluid in general relativity. A number of previously known classes of solutions has been rediscovered in the process. The method indicates the possibility of constructing a plethora of new physically significant models of relativistic stellar interiors with equations of state fairly applicable to the case of extremely compressed stars. To emphasize our point we have derived two new classes of solutions and discussed their physical importance. From the solutions of these classes we have constructed three causal interiors out of which in two models the outward march of pressure, density, pressure-density ratio and the adiabatic sound speed is monotonically decreasing.

1. Introduction

There have been a few attempts to obtain parametric classes of exact solutions of Einstein’s field equations describing the interior field of perfect fluid balls in equilibrium (Tolman, 1939; Wyman, 1949; Kuchowicz, 1968, 1970; Pant and Sah 1982, 1985; Pant and Pant, 1993, 1993a,b). The importance of a parametric class of solutions over an ordinary solution lies in the flexibility of the associated parameter which brings out various models of relativistic star with physically realizable fluid properties. Moreover, given a class of solutions, by imposing realistic conditions on the parameter one may weed out unphysical as well as insignificant solutions. For instance, not all solutions in a class would correspond to causal models and therefore the causality principle shall limit the range of the associated parameter.

Methods have been suggested to obtain classes of solutions but the scope of such attempts is limited to generalize known particular solutions (Leibovitz, 1969; Goldman, 1978, Matase and Whitman, 1980; Whitman, 1983). In this paper we present a method of integrating Einstein’s field equations which results into a plethora of parametric classes of physically sound solutions. The paper indicates as to how a host of new possibilities may emerge for the meaningful integration of the field equations.

2. Field Equations and Method of Integration

In canonical coordinates the metric of a static, spherically symmetric field is

$$ds^2 = -e^{\lambda(r)} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) + c^2 e^{\nu(r)} dt^2. \quad (1)$$

It follows that the field equations of general relativity for a ball of perfect fluid with pressure \( p(r) \) and density \( \rho(r) \) are (Tolman, 1939)

\[
\frac{8\pi G}{c^4} p = e^{-\lambda} \left( \frac{1}{r} \frac{d\nu}{dr} + \frac{1}{r^2} \right) - \frac{1}{r^2}, \tag{2}
\]

\[
\frac{8\pi G}{c^2} \rho = e^{-\lambda} \left( \frac{1}{r} \frac{d\lambda}{dr} - \frac{1}{r^2} \right) + \frac{1}{r^2}, \tag{3}
\]

\[
\frac{d}{dr} \left( \frac{e^{-\lambda} - 1}{r^2} \right) + \frac{d}{dr} \left( \frac{e^{-\lambda} \frac{d\nu}{dr}}{2r} \right) + e^{-\lambda-\nu} \frac{d}{dr} \left( \frac{e^\nu \frac{d\nu}{dr}}{2r^2} \right) = 0. \tag{4}
\]

By allowing one of the two field variables \( \lambda \) and \( \nu \) as some known function of \( r \) the equation (4) transforms into a form which on integration determines the metric (1) completely, the fluid parameters are then calculated from (2) and (3).

By subjecting (4) to the transformation

\[
U = r^m e^{m\nu/2}, \quad V = e^{-\lambda}, \tag{5}
\]

\( m \) being a non-zero arbitrary constant, we obtain a linear differential equation in \( V \):

\[
\frac{dV}{dr} - 2 \left\{ \frac{d}{dr} \log \left( \frac{r^3 U^{1-1/m}}{dU/dr} \right) - \left( \frac{2mU}{r^2 dU/dr} \right) \right\} V = -\frac{2mU}{r^2 dU/dr}. \tag{6}
\]

Integration yields

\[
e^{-\lambda} = V = \left[ \frac{r^6 U^2(1-1/m)}{(dU/dr)^2} \right] \times \]

\[
\times \left[ A - 2 \int \frac{m dU}{r^8} U^{(-1+2/m)} e^{\int \left[ \frac{4mU}{r^2 (dU/dr)} \right] dr} \right] \times \]

\[
\times e^{-\int \left[ \frac{4mU}{r^2 (dU/dr)} \right] dr}, \tag{7}
\]

where \( A \) is another arbitrary constant. Our aim is to explore the possibilities of choosing \( U \) such that the right hand side of (7) becomes integrable. A set of possibilities arises if one assumes for \( e^{\int \left[ \frac{4mU}{r^2 (dU/dr)} \right] dr} \) some algebraic function of \( r, U \) and \( dU/dr \). In this paper we assume

\[
e^{\int \left[ \frac{4mU}{r^2 (dU/dr)} \right] dr} = r^l \left( \frac{dU}{dr} \right)^n, \tag{8}
\]

\( l \) and \( n \) being arbitrary constants. Equation (8) results into a second order homogeneous equation in \( U \):

\[
nr^2 \frac{d^2 U}{dr^2} + lr \frac{dU}{dr} - 4mU = 0. \tag{9}
\]