About the Possibility of Increasing the Superconducting Transition Temperature by the Phonon Mechanism

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The equations for the energy gap and for the critical temperature $T_c$ are investigated in the case of superconductors with strong coupling, with regard for the dependence of the interaction constant on the phonon frequency. The effective interaction constant $\lambda$, which determines $T_c$ in the Einstein model and delta-function phonon peak model, is calculated, and its dependence on the ordinary interaction constant, phonon frequency and momentum, Fermi momentum, and other parameters is determined. It is shown that the maximum possible value of $\lambda$ is unity. A solution is found for the critical temperature equation in the case of a delta-function peak and the Einstein model. It is demonstrated that in the latter case $T_c$ has a sharp maximum as a function of the electron density.

1. THE GAP EQUATION

In solving the problem of the possibility of increasing the critical transition temperature of superconductors under the ordinary phonon coupling mechanism one must clearly rely on the real Fröhlich model, which takes direct account of the interaction between electrons and phonons (see Ref. 1). In the present paper we calculate the superconducting transition temperature on the basis of the exact solution of the Eliashberg equations, which describes the Fröhlich model for any degree of electron-phonon coupling. 2

The solution of the Dyson equations in the presence of pairing, i.e., of the Gor’kov equations in $p$ representation at finite temperatures, has the form

$$G = (G_0^{-1} - \Sigma_1-) / \Omega$$
$$F \equiv F^+ = \Sigma_2^\ast / \Omega$$
$$\Omega = (G_0^{-1} - \Sigma_1)(G_0^{-1} - \Sigma_1-) + |\Sigma_2|^2$$

where $G$ and $F$ are the Green function and $F$ function, $G_0 = (i\omega_n - \xi_0)^{-1}$ is the free-particle Green function, $\Sigma_1$ and $\Sigma_2$ are the ordinary and pairing self-energy parts, the minus sign indicates the replacement of $\omega_n$ by $-\omega_n$, $\omega_n = (2n + 1)\pi T$, and $\xi_0$ is the free-particle energy measured relative to the Fermi level. Our primary concern will be the behavior of the $F$ function in a layer slightly thicker than the
Debye frequency \( \omega_D \) next to the Fermi surface; we can limit then ourselves to the pole part of the Green function for a normal metal. This is equivalent to the representation of \( \Sigma_1 \) at low frequencies \( (\omega_n \lesssim \omega_D) \): \( \Sigma_1(\mathbf{p}, \omega_n) \approx \Sigma_{10}(\mathbf{p}) - i \omega_n \). Then from (1) we find

\[
G = a(-i\omega_n - \xi)/\tilde{\Omega} \quad F = a\Delta/\tilde{\Omega}, \quad a = (1 + b)^{-1}
\]

(2)

Here, \( \Delta = a\Sigma_2 \), \( a = v_0/v_0 = m/m_{\text{eff}} \) determines the renormalization of the electron velocity on the Fermi surface,\(^3\) and \( \Delta \) is the frequency-dependent energy gap; \( \Sigma_2 \equiv \Delta(\omega_n)\).\(^2,3\)

The equation for \( \Sigma_2 \) has the familiar form\(^2,4\)

\[
\Sigma_2(\mathbf{p}, \omega_n) = \frac{T}{2(2\pi)^3} \sum_{\mathbf{n}} \int \frac{d\mathbf{k} F(p, k; p - k)}{\cos(q)} \cdot F(k, \omega_n)
\]

(3)

\[
D(\mathbf{q}, \omega_n) = \omega^2(\mathbf{q})/[\omega^2(\mathbf{q}) + \tilde{\omega}_n^2]
\]

where \( \Gamma \) is the three-tailed vertex describing the interaction of electrons with phonons and, up to terms of order \((m/M)^{1/2}\), is equal to \( \Gamma_0 \) (\( \Gamma \) in the zeroth approximation), where \( m \) and \( M \) are the electron and ion masses,\(^4\) \( D \) is the phonon Green function, and \( \omega(\mathbf{q}) \) is the phonon frequency. The renormalization of \( \omega(\mathbf{q}) \) may be neglected.\(^1,5\) The function \( \Gamma \) is customarily replaced by a constant; actually, however, as we know from the theory of conduction of metals, it is inversely proportional to \( \omega(q) \); the latter fact was brought to attention by McMillan\(^7\) (see, also, Ref. 8). If we also take account of the dependence of \( \Gamma \) on the phonon momentum \( \mathbf{q} \) and on the direction of the phonon polarization vector \( \mathbf{e}_j \) (where \( j \) is the order number of the phonon branch), then

\[
\bar{\Gamma}^2_j \equiv \nu_0 \Gamma_j^2 = \xi_j(\mathbf{q})u^2q^2/\omega^2(q)
\]

(4)

where \( u \) is the velocity of sound, \( \nu_0 \) is the density of states on the Fermi surface, and \( \xi_j \) is the product of \( (e_j, \mathbf{q})/\mathbf{q} \) and a function of \( q \) that varies only slightly for small \( q \); according to Ref. 9 this function has the form \( \kappa^4/(\kappa^2 + q^2)^2 \) (where \( \kappa^{-1} \) is the Debye radius). The quantity \( \gamma(\mathbf{q}) \) can also involve terms associated with Umklapp processes. In this event, of course, \( \gamma(\mathbf{q}) \) can turn out to be a function that no longer depends only slightly on \( \mathbf{q} \). For a purely acoustical dispersion law and \( \gamma = 1 \) we have \( \bar{\Gamma}^2 = \xi \). Let us substitute (4) into (3) and integrate over the azimuthal angle, replacing the integration over \( \mathbf{k} \) by integration over \( \xi \) (it is important to bear in