CONSTITUTIVE EQUATIONS OF PLASTICITY OF A TWO-PHASE HARDENING MEDIUM

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In this work, a continual model of a two-phase composite medium with hardening components is constructed on the macrolevel. Fields of microstresses and microstrains in a heterogeneous medium, determined by transformation or residual phase strains, are examined. In addition, constitutive equations are obtained for plastic deformation in increments on the macrolevel using the equation for averaging plastic strains.

1. We examine a two-phase elastoplastic composite medium (characteristic volume $V=V_{(1)}+V_{(2)}$, and $v_i=V_{(i)}/V$ are the coefficients of the volume content of the components). We shall assume that the homogeneous boundary conditions [1, 2], determined by a constant stress $\sigma$ or strain $\varepsilon$ tensor are specified on the surface of the characteristic volume. By $<(...)>$ denote the averaging operation with respect to the characteristic volume $V$, and $<(...)>(i)$ — with respect to the phase volume $V_{(i)}$. Subsequently, all tensor quantities with the asterisk denote the local microvariables inside the characteristic volume, those with the index $(i)$ denote their mean values with respect to the corresponding phase, and those without additional indices denotes the macrovariables or effective characteristic of the composite medium.

The following conditions are satisfied in the elastic condition of the material (the colon denotes the convolution operation with respect to two indices)

$$
\sigma(x) = \begin{cases}
A_{(1)}(x) : \sigma(x), & x \in V_{(1)}; \\
A_{(2)}(x) : \sigma(x), & x \in V_{(2)};
\end{cases}
$$

$$
\varepsilon(x) = B_{(1)}^e(x) : \varepsilon(x),
$$

where $B_e^{(i)}(x)$ and $D_e^{(i)}(x)$ are the tensors of concentration of elastic stresses and strains. After substituting (1.3) into (1.1) and averaging out or taking into account Eq. (1.2), we obtain

$$
A' = v_1 A_{(1)} : B_{(1)}^e + v_2 A_{(2)} : B_{(2)}^e; \\
C' = v_1 C_{(1)} : D_{(1)}^e + v_2 C_{(2)} : D_{(2)}^e.
$$

It is usually not possible to obtain an exact solution of the elastic problem for the homogeneous medium $B_e^{(i)}(x)$ and $D_e^{(i)}(x)$. However, to compute the effective elastic constants it is sufficient, as indicated by Eq. (1.4), to know the mean tensors $B_{(1)}^e$ and $D_{(1)}^e$ with respect to the phase. Estimates of these tensors can be obtained by solving different modeling problems [3, 4]. In addition, the mean tensors of stress concentration can be expressed by means of the elastic constant of the composite and its components [5]:
\[ v_1 B_1 = (\mathcal{A}_1^2 - \mathcal{A}_2^2)^{-1} : (\mathcal{A}_1 - \mathcal{A}_2) ; \quad v_2 B_2 = (\mathcal{A}_3^2 - \mathcal{A}_1^2)^{-1} : (\mathcal{A}_3 - \mathcal{A}_1). \] (1.5)

2. The stresses and strains in a heterogeneous medium can be caused by both mechanical loading and certain transformation or residual strains of the components. Examples include temperature strains or strains in phase transition.

We shall write, in a more general form, the equations (1.1) of Hooke's law

\[ \varepsilon = \mathcal{A} : \sigma + \alpha + \beta; \quad \mathcal{A} = C : \varepsilon + \beta. \] (2.1)

where \( \alpha \) and \( \beta \) are the transformation strains and stresses, respectively. Consequently, the distribution of stresses and strains inside the characteristic volume can be written in the form

\[ \varepsilon = \mathcal{A}_1 : \sigma_1 + \mathcal{A}_2 : \sigma_2 + \mathcal{A}_3 : \sigma_3; \quad \mathcal{A}_1 = C_1 : \varepsilon + \beta_1; \quad \mathcal{A}_2 = C_2 : \varepsilon + \beta_2; \quad \mathcal{A}_3 = C_3 : \varepsilon + \beta_3. \] (2.2)

and the constitutive equation of the macrolevel in the form

\[ \varepsilon = \mathcal{A} : \sigma; \quad \sigma = C : \varepsilon + \beta. \] (2.3)

We shall formulate the problem of determining the tensors \( \mathcal{D}_1(x) \), \( \mathcal{D}_2(x) \), \( \mathcal{B}_1(x) \), \( \mathcal{B}_2(x) \) and the mean characteristics \( \mathcal{A} \) and \( \beta \), assuming that the solution of the elastic problem \( \mathcal{D}_e(x) \) and \( \mathcal{B}_e(x) \) is available. We shall use the result for homogeneous fields obtained in [6]. If the components of the two-phase composite undergo transformation strains \( \alpha(i) \), \( i = 1, 2 \), there is always a homogeneous external stress field \( \sigma^0 \) which when applied results in the microstresses and microstrains on the surface of the characteristic volume of the film becoming homogeneous:

\[ \varepsilon = \varepsilon; \quad \sigma = \sigma^0. \]

When the elastic constants of the components completely differ and \( (A^e_1 - A^e_2) \) is a nondegenerate tensor, the result for \( \sigma^0 \) is very simple:

\[ \sigma^0 = (\mathcal{A}_1^2 - \mathcal{A}_2^2)^{-1} : (\alpha(2) - \alpha(1)). \] (2.4)

It can be verified that in this case \( v_x = v_x(2) \). Adding the solution of the type (1.3) for \( \sigma = -\sigma_0 \), we obtain the external surface free from loading and a field of microstresses caused by phase strains

\[ \alpha = [I - B_e(x)] : (\mathcal{A}_1^2 - \mathcal{A}_2^2)^{-1} : (\mathcal{A}_2 - \mathcal{A}_1). \] (2.5)

Comparing (2.5) with (2.2) we can write

\[ B_e(x) = (-1)^i [I - B_e(x)] : (\mathcal{A}_2^2 - \mathcal{A}_1^2)^{-1} : \mathcal{A}_e. \] (2.6)

where \( I \) is the unit tensor of the force rank.

As indicated by Eq. (2.3)

\[ \alpha = < \varepsilon > \quad \text{for} \quad \sigma = 0. \] (2.7)

Substitution of (2.1) and (2.5) into (2.7) gives

\[ \alpha = v_1 \alpha_1 + v_2 \alpha_2 + (v_1 \mathcal{A}_1^2 + v_2 \mathcal{A}_2^2 - \mathcal{A}^2) : (\mathcal{A}_1^2 - \mathcal{A}_2^2)^{-1} : (\mathcal{A}_2 - \mathcal{A}_1). \] (2.8)