STOCHASTIC APPROACH TO CAUCHY SYSTEM OF CHANDRASEKHAR'S PLANETARY PROBLEM WITH SPECULAR REFLECTOR

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Abstract. In the present paper, assuming that the radiative transfer in turbid slab bounded below by the imperfect specular reflector is the Markovian process, and starting with the modified Chapman-Kolmogorov equation for a uniform monodirectional illumination, we obtained the Riccati-type of the nonlinear integro-differential equation governing the angular distribution of the radiation emergent from the slab. In other words, without solving the transfer equation, the two-point boundary value problem is reduced to an initial-value problem, by which the numerical simulation is readily performed with the aid of the high-speed digital computer.

1. Introduction

The theory of spectral lines in stellar and planetary atmospheres plays an important role in astrophysics and meteorology, inquiring into the physical processes in the multiple scattering of photons. The equation of radiative transfer is the linearized Boltzmann equation governing the photon diffusion process in turbid media. Based on the invariance principles, Ambarzumian (1943) has solved the standard diffuse reflection problem in turbid slab without starting with the transfer equation. Subsequently, Chandrasekhar (1950) extended physically and mathematically the Ambarzumian's invariance principles and applied them to the miscellaneous astrophysical problems. Thereafter, Bellman et al. (1963) have developed the invariant imbedding procedure for the radiative transfer in slab geometry. Furthermore, van de Hulst (1963) has dealt with the diffuse reflection problem in turbid slab by means of the adding procedure.

All of the above authors have not explicitly referred to the transfer equation. On the other hand, Kourganoff (1952), Sobolev (1956), and Busbridge (1960) have rigorously solved the transfer equation in turbid slab, whose solution has been proved to reduce to that given by the above authors.

Whereas in the standard problems the radiative transfer in free atmosphere has been discussed, Chandrasekhar (1950) has dealt with the diffuse reflection of radiation from an atmosphere which absorbs and scatters an incident solar radiation, being bounded by the diffuse reflector. Thereafter, Sobolev (1956), Casti et al. (1969), Ueno and Mukai (1972), Kagiwada (1974), Kagiwada et al. (1975), Matsumoto and Ueno (1976), have dealt with the angular distribution of radiation emergent from the atmosphere bounded by the specular reflector.

An aim of our present paper is to demonstrate a straightforward physical derivation of the Cauchy system for the diffuse reflection problem in turbid slab bounded by the specular reflector from the Chapman-Kolmogorov equation, assuming that the photon diffuse reflection by turbid slab is the Markovian process. In other words, without either solving the transfer equation or referring to the invariance principles, an initial-value solution of the two-point boundary value problem is found by means of the modified Chapman-Kolmogorov equation, which governs the photon diffusion process (cf. Ueno, 1957, 1958, 1960, 1962, 1965, 1974, 1984; Bharucha-Reid, 1960).

2. Basic Equations

Consider a plane-parallel, homogeneous, and anisotropically scattering atmosphere of the optical thickness \( x \) bounded below by a specular reflector. Denote the albedo for single scattering by \( \lambda \). Within the slab, the imperfect Rayleigh scattering is assumed to prevail everywhere. The upper surface allows the passage of radiation energy in any direction. Let uniform parallel rays of net flux \( F \) be incident on the top of the atmosphere at \( t = x \), the angle of incidence being \( \arccos(u) \). The polar angle \( \arccos(v) \) of a pencil of radiation is measured from the normal at \( x \). We shall confine our attention to the azimuth-independent terms of partially-polarized light, i.e., the specific intensity of the components parallel or normal to the vertical plane, through the direction of propagation.

Let \( p(v, v') \) denote the two-by-two phase matrix in accordance with Rayleigh scattering.

The specular reflector is characterized by the function \( A(v) \), \( 0 \leq A \leq 1 \), which represents the probability that a photon impinging on the bottom surface with angle of incidence \( \arccos(v) \) is specularly reflected. Let the source function \( J \) be the total rate of production of scattered energy per unit volume per unit solid angle at the optical height \( t \) above the bottom reflecting surface.

Then, the two-by-two source matrix \( J \) is given by (cf. Ueno, 1974)

\[
J(t, v, u; x) = \frac{\lambda}{4} P(v, u) F(e^{-(x-t) / u} + A(u) e^{-(x+t) / u} + \frac{\lambda}{2} \int_{-1}^{1} P(v, w) J(t, w) \, dw +
\]

\[
+ \frac{\lambda}{2} \int_{0}^{1} A(w) P(v, w) e^{-\omega w} \, dw \int_{0}^{x} J(y, w, u; x) e^{-\gamma \omega} \frac{dy}{w}, \quad (1)
\]

where \( I(t, v) \) represents one-column intensity matrix.

Starting with Equation (1) and inverting the order of integration with respect to \( w \) and \( y \), we have

\[
J(t, v, u; x) = \frac{\lambda}{4} P(v, u) F(e^{-(x-t) / v} + A(u) e^{-(x+t) / u} +
\]