The physical characteristics radius, mass, mean density, gravitational potential and acceleration, gravitational and internal energy are presented with the aid of the gamma function for $N$-dimensional, radially-symmetric polytropes. The virial theorem with external pressure is derived in the relativistic limit, with Newtonian gravitation still valid. The gravitational energy of polytropes obeying the generalized Schuster–Emden integral is shown to be finite. Finiteness of mass and radius is discussed for the cases of practical interest $N = 1$ (slab), $N = 2$ (cylinder), and $N = 3$ (sphere). Uniform contraction or expansion of $N$-dimensional polytropes is considered in the last section.

1. Introduction

The physical characteristics of polytropes have been widely quoted in the astrophysical literature (e.g., Emden, 1907; Chandrasekhar, 1939; Viala and Horedt, 1974a; Kimura and Liu, 1978; Abramowicz, 1983), but the relevant equations are valid only for some particular values of the geometric and polytropic index (e.g., Chandrasekhar, 1939; Viala and Horedt, 1974a), or quote only the most simple physical characteristics as mass, radius, and mean density (e.g., Abramowicz, 1983). The importance of the work of Kimura and Liu (1978) is reduced by the fact that they employ additional transformations for the canonical Lane–Emden variables. So I have decided to present this contribution, since besides some new findings, it derives and summarizes systematically the equations spread out over the literature.

2. Pressure, Density, Temperature, Radius, Volume, Mass, and Mean Density

Euler’s equations of motion of a nonmagnetic, viscosity-free fluid under the action of gravitational and pressure forces are of the form

$$\rho \frac{dv}{dt} = -\nabla P + \rho \nabla \Phi_i,$$  \hspace{1cm} (2.1)

where $\rho$ denotes the density; $dv/dt$, the substantial derivative of the velocity; $P$, the hydrostatic pressure of the fluid; and $\Phi_i$, the internal Newtonian gravitational potential. For hydrostatic equilibrium ($v = 0$), Equation (2.1) writes in a space with radial symmetry

$$dP/dr = \rho \frac{d\Phi_i}{dr},$$  \hspace{1cm} (2.2)

where $r$ is the radial distance.
The radially-symmetric form of Poisson’s equation in $N$-dimensional space is according to Equation (A.15)

\[
\left(1/r^{N-1}\right) d\left(r^{N-1} d\Phi/r dr\right)/dr = -4\pi G\rho .
\]

(2.3)

We write the pressure and density of the fluid in terms of the Lane-Emden variables $\xi$ and $\theta = \theta(\xi)$, (e.g., Chandrasekhar, 1939)

\[
\rho = \rho_0 \theta^n ; \quad P = K \rho^{1+1/n} = P_0 \theta^{n+1} , \quad (n \neq -1, \pm \infty) ,
\]

(2.4)

\[
\rho = \rho_0 \exp(-\theta) ; \quad P = K \rho = P_0 \exp(-\theta) , \quad (n = \pm \infty) ,
\]

(2.5)

where $n$ denotes the polytropic index and $K$ the polytropic constant.

It should be noted that most subsequent equations are valid also for an arbitrary point inside a polytrope.

Equations (2.4) and (2.5) are inserted into Equation (2.2). After integration, the result is inserted into Poisson’s equation (2.3), yielding the well known Lane-Emden equations

\[
x^{1-N} d(x^{N-1} d\theta/dx)/dx = \theta'' + (N-1)\theta'/x = F - \theta'',
\]

(0' = d\theta/dx ; \theta'' = d^2\theta/dx^2 ; N = 1, 2, 3, ..., \ n \neq -1, \pm \infty) ,

(2.6)

\[
x^{1-N} d(x^{N-1} d\theta/dx)/dx = \theta'' + (N-1)\theta'/x = \exp(-\theta),
\]

(\ N = 1, 2, 3, ..., n = \pm \infty) .

(2.7)

The upper sign on the left-hand side of Equation (2.6) corresponds always to values of the polytropic index $-1 < n < \infty$, and the lower one to $-\infty < n < -1$.

To get Equations (2.6) and (2.7) we have inserted for the radial distance according to

\[
r = x = \left[ \pm (n + 1)K/4\pi G\rho_0^{1-1/n} \right]^{1/2} x , \quad (n \neq -1, \pm \infty) ,
\]

(2.8)

\[
r = x = (K/4\pi G\rho_0^{1/2})^{1/2} x , \quad (n = \pm \infty) ;
\]

(2.9)

$r$ is the radial distance from the symmetry plane of a polytropic slab ($N = 1$), from the symmetry axis of a polytropic cylinder ($N = 2$), and from the centre of a $N$-dimensional sphere ($N \geq 3$).

The function $\theta$ satisfies the well-known initial conditions

\[
\theta(0) = 1 ; \quad \theta'(0) = 0 , \quad \text{if} \quad n \neq 1, \pm \infty ,
\]

(2.10)

\[
\theta(0) = 0 ; \quad \theta'(0) = 0 , \quad \text{if} \quad n = \pm \infty .
\]

(2.10)

If the polytrope obeys the initial conditions (2.10) for the Lane-Emden variables $\xi$ and $\theta$, then $P_0$ and $\rho_0$ from Equations (2.4) and (2.5) are just equal to the pressure and density at radial distance $r = 0$. 