EXACT SOLUTION OF AXIALLY SYMMETRIC RAYLEIGH SCATTERING TRANSPORT EQUATION
BY LAPLACE TRANSFORM AND WIENER-HOPF TECHNIQUE AND NEW EXPRESSION OF $H_i$ AND $H_r$ FUNCTION

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Abstract. An exact solution of the transport equation in radiative transfer for an axially symmetric Rayleigh scattering problem in semi-infinite planetary atmosphere both for emergent intensity and intensity at any optical depth has been derived with the help of the Laplace transform and the Wiener–Hopf technique, and by use of the constancy of net flux. Chandrasekhar's results for emergent intensity have been verified. New expressions for the $H_i$ and $H_r$ functions have been obtained.

Chandrasekhar (1946, 1947) considered an axially-symmetric Rayleigh scattering problem of radiative transfer with a constant net flux. He applied the discrete ordinate method to get the $n$th approximated emergent intensity and the intensity at any optical depth in a semi-infinite plane-parallel atmosphere. An exact expression of emergent intensity had been obtained by letting $n$ tend to infinity. Siewert and Frayley (1967), using the basic technique of normal mode expansions that were initiated by Case (1960), were able to solve exactly and rigorously the equation of radiative transfer formulated by Chandrasekhar (1946) both for emergent intensity and intensity at any optical depth. Domke (1971) had treated Chandrasekhar's problem of conservative Rayleigh scattering in a semi-infinite atmosphere. He expanded the scattering matrix suitability to introduce scalar source functions which depend only on the optical depth, and obtained independent Wiener–Hopf equations for these source functions which are solved by the Sobolev method.

In this paper the problem has been solved with the help of the Laplace transform and the Wiener–Hopf technique. The transport equation is subjected to the Laplace transform to get an integral equation for the emergent intensity. This integral equation has been divided into two parts, which is the usual procedure in the Wiener–Hopf technique. One part gives the emergent intensity while the other gives the unknown constants considering the constancy of the net flux. The Laplace transform of intensity at any optical depth has been expressed in terms of the $H_i$ and $H_r$ functions of Chandrasekhar (1950). The intensity at any optical depth has been obtained by inversion. For getting the residues at the poles of the integrand and inverting, the expression

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for the $H$-function, obtained by Dasgupta (1977), has been taken into consideration.
Finally, the intensity in the $l$ and $r$ directions for the positive direction parameter $u$
has been derived using the properties of the $H$-functions on both sides of the singular
line. The intensity in the $l$ and $r$ directions for the negative direction parameter $u$ has
been derived using the Plemelj's formulae in terms of Cauchy's principal value. The
expressions for the $H_l$ and $H_r$ functions have been derived by equating the intensity
at $t = 0$ with that of the emergent intensity.

Let us consider an axially symmetric radiation field which scatters radiation in
accordance with Rayleigh's law in a semi-infinite atmosphere with no incident radia-
tion. Four Stokes parameters, $I_l(t, u)$, $I_r(t, u)$, $U(t, u)$ and $V(t, u)$ – the four com-
ponents of a column matrix $I(t, u)$ – will suffice to characterize the radiation field.
As the radiation field is axially symmetric, the two parameters $U(t, u)$ and $V(t, u)$
will vanish. The scalar equation for $I_l(t, u)$ and $I_r(t, u)$ (cf. Chandrasekhar, 1950, Section
68) are

$$u \frac{dI_l(t, u)}{dt} = I_l(t, u) - \frac{3}{4}[2(I_{00}(t) - I_{12}(t)) + u^2(3I_{12}(t) -
- I_{00}(t) + I_{r0}(t))]$$

$$u \frac{dI_r(t, u)}{dt} = I_r(t, u) - \frac{3}{4}(I_{12}(t) + I_{r0}(t)), (2)$$

where $t$ is the optical depth and $u$ is the direction parameter, and

$$I_{km}(t) = \int_1^{-1} u^m I_k(t, u) \, du, \quad k = l, r, \quad m = 0, 1, 2. (3)$$

The boundary condition for solving Equations (1) and (2) is

$$I_k(0, -u) = 0, \quad 0 \leq u < 1, \quad 0 \leq u < 1, \quad (4a)$$

$$I_k(t, u) \exp(-t/u) \rightarrow 0 \quad \text{when} \quad t \rightarrow \infty, \quad |u| < 1, \quad k = l, r. \quad (4b)$$

Let us define $I^*(s, u)$, the Laplace transform of $I(t, u)$, as

$$I^*(s, u) = \int_0^{\infty} I(t, u) \exp(-st) \, ds, \quad \Re s > 0. (5)$$

Let us set

$$I_{km}^*(s) = \frac{1}{2} \int_1^{-1} I_k^*(s, u) u^m \, du, (6a)$$

$$= s \int_0^{\infty} I_{km}(t) \exp(-st) \, ds. (6b)$$

Subjecting the Laplace transform as defined in Equation (5) to Equations (1) and (2),
we obtain

$$(us - 1) I_k^*(s, u) = usI_k(0, u) - \frac{3}{4}[2(I_{00}^*(s) - I_{12}^*(s)) + u^2(3I_{12}^*(s) -
- 2I_{00}^*(s) + I_{r0}^*(s))]$$

(7)