GENERAL VARIATIONAL METHOD FOR MILNE'S INTEGRAL EQUATION

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Abstract. In this paper, we establish a general variational method for Milne's integral equation. Moreover, a recursive computational algorithm for the method is also constructed. To illustrate the precision of the method as well as the computational algorithm, a numerical example for \( N = 20 \) is given.

1. Introduction

The problem of isotropic scattering of radiation with constant net flux in a plane-parallel atmosphere, is governed by Milne's integral equation

\[
q(y) = \frac{1}{2} \int_0^\infty q(y') E_k(|y - y'|) \, dy' + \frac{1}{2} E_0(y), \quad (1.1)
\]

with Hopf's condition (cf., e.g., Kourganoff, 1952; p. 64)

\[
q(y) \rightarrow \text{constant as } y \rightarrow \infty, \quad (1.2)
\]

where \( E_k(t) \) is the exponential integral of the \( k \)th order, defined by

\[
E_k(t) = \int_1^\infty \frac{e^{-tv}}{v^k} \, dv. \quad (1.3)
\]

Some authors in the field of radiative transfer have proposed variational methods for the solution of Equation (1.1). These may be classified into two main types: type A is characterized by a non-linear variational system of equations for the unknown parameters; while type B is characterized by a linear variational system. When the system associated with type A is of order greater than 2, it is difficult to apply in practice. Moreover, all the methods of type B suffer in common from serious practical inconveniences:

(a) A lack of general expressions for the equations involved. This is because the authors using type B (e.g., Huang, 1953) restrict themselves from the beginning to a few numbers of the basic functions for their trial approximate solution. In the absence of general expressions, the computations cannot be reduced to automatic work.

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(b) The functions involved are so complicated that a general computational algorithm is seldom, if ever, available because of extensive computer storage requirements and long execution times.

Due to the above practical difficulties it is not very surprising that no serious attempts have been made to use the methods of each type (as they stand in the literature) for constructing a general variational method for Equation (1.1) with a flexible computational algorithm. Thus, a general and flexible variational method must now be developed with the following requirements:

1. The resolvent functional is to be selected so that the variational system of equations for the unknown parameters is linear.
2. The trial approximate solution is to be considered generally from the beginning, and should conform to the required condition given by Equation (1.2).
3. The basic equations governing the variational method must be expressed in explicit general forms.
4. Each of the functions included in the basic equations (of requirement (3)) should be analysed and put into proper computational form.

The aim of the present work will be to develop a general variational method for the solution of Equation (1.1) based on a flexible computational algorithm.

For the first requirement, we shall use Huang's (1953) resolvent functional of the form

$$I(q) = q(y)q(y) - \frac{1}{2} \int_0^\infty q(y')E_1(|y - y'|)\,dy' \,dy - \int_0^\infty q(y)E_0(y)\,dy,$$

(1.4)

for which he has shown that it will be stationary for the actual solution of Equation (1.1).

For the second requirement, the general trial approximate solution is chosen as

$$q(y) = c_1 + \sum_{k=2}^N c_k E_k(y),$$

(1.5)

where $E_k(y)$ is defined by Equation (1.3) and the $c_k$'s are parameters to be determined. Evidently, Equation (1.5) satisfies the required condition that $q(y) \to$ constant ($= c_1$) as $y \to \infty$.

The third requirement will be the subject of Section 2, while Sections 3 and 4 are devoted to the fourth requirement.

2. Formulation of the Basic Equations

Before starting the analysis, we find it profitable to introduce the abbreviations

$$F_{nm} = F_{mn} = \int_0^\infty E_n(t)E_m(t)\,dt$$

(2.1)