FRAME TRANSFORMATION OF SPHERICAL-HARMONIC COEFFICIENTS OF DIFFERENTIAL PARTICLE INTENSITY

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Abstract. We present a general second-order-correct frame transformation on spherical-harmonic coefficients of differential particle intensity. The transformation, valid for relativistic particles as well, provides a clear view of the Compton-Getting effect. It shows explicitly how each transformed harmonic coefficient depends on a subset of the original harmonic coefficients. The general expression for the first-order Compton-Getting vector anisotropy is derived and interpreted. In addition, we show how the new transformation allows one to simplify a current procedure for determining the directional intensity in a comoving frame. This involves the directional particle data measured on a spacecraft.

1. Introduction

The Compton-Getting effect is a result of the transformation of the differential particle intensity from one frame of reference to another at the constant relative velocity \( W \) (Compton and Getting, 1935; Gleeson and Axford, 1968). Suppose that in each frame of reference, the differential intensity is expressed as a series of orthogonal-spherical harmonics with coefficients dependent on momentum magnitude (see Equations (1) and (2)). The full Compton-Getting effect is manifest in the explicit relations that express each spherical-harmonic coefficient of the intensity in the one frame in terms of the spherical-harmonic coefficients of the intensity and their derivatives in the other frame. This statement is a general and useful definition of the classical concept.

In this paper we show how to determine the Compton-Getting effect. We derive the general transformation formulae for all harmonic coefficients which is correct to the second order in \( W/v \), where \( v \) is the particle speed. These formulae are also valid for relativistic particles.

The first-order Compton-Getting streaming (i.e., the first-order first harmonics) had been considered in many studies of galactic cosmic rays and solar particles before its correct expression for near-isotropic distribution was derived by Gleeson and Axford (1968) and Forman (1970a). This expression has been widely exploited in the interpretation and modeling of the anisotropy of solar and galactic particles (e.g., Forman, 1970b; Ng, 1971; Forman and Gleeson, 1975). However, in their derivations, Gleeson and Axford, and Forman, assumed an isotropic or near-isotropic distribution, considered only the zeroth and first harmonics, and neglected terms of order \( (W/v)^2 \). Thus their transformations are inapplicable for highly anisotropic distributions and/or particles of very low energy, say below \( \sim 200 \text{ keV} \text{ nucl}^{-1} \). In recent years, the studies of highly anisotropic solar-flare particles and low-energy particles associated with interplanetary shocks and planetary bow shocks have attracted considerable interest. They
require the consideration of higher harmonics and higher-order effects. The general second-order correct transformation on spherical-harmonic coefficients, to be derived in this paper, should facilitate such studies, at least for particles with energy \( \geq 10 \text{ KeV nucl}^{-1} \).

Balogh et al. (1973) obtained a second-order correct non-relativistic transformation from an isotropic distribution in one frame to the zeroth through second harmonics in another frame. They also derived a first-order correct transformation for an anisotropic distribution. This distribution contained terms up to the first harmonics. Their results, however, are not expressed in terms of spherical harmonics and the interpretation is not straightforward. Ng (1984) derived a second-order-correct relativistic transformation which expresses each spherical-harmonic coefficient of the differential intensity in the spacecraft frame in terms of the harmonic coefficients of an arbitrary gyrotropic differential intensity in a comoving frame. The above transformations may be regarded as special cases of the general direct transformation which will be considered in this paper.

Many sophisticated methods have been devised to determine the differential intensity in a comoving frame (e.g., the solar wind frame) from the sectored counting rates measured on a spacecraft (e.g., Ipavich, 1974; Gold et al., 1975; Erdős, 1981; Sanderson et al., 1985; and references given therein). The effect of the frame transformation has been incorporated into these methods via the Lorentz invariance of the phase-space distribution function and the transformation of the individual particle momenta. The result is considerable complexity in the fitting procedures (see Section 5). An important application of the direct transformation formulae is the decoupling of the experimental determination of the differential intensity from the theoretical consideration of frame transformation effects.

In Section 2, we define the problem and explain the symbols used. The transformation is then derived in Section 3, with more details relegated to Appendix A. In Section 4, we express the differential mean intensity and streaming in the spacecraft frame in terms of the quantities in the comoving frame, and derive the most general expression for the first-order Compton–Getting vector anisotropy, which is, in general, not aligned with the transformation velocity \( \mathbf{W} \). In Section 5, we show how the new transformation allows one to simplify a current procedure for determining the directional differential intensity in a comoving frame. A summary and concluding remarks are given in Section 6.

2. The Problem and the Coordinate System

Suppose the differential intensity with respect to momentum (or rigidity) is expressed in the spacecraft frame by

\[
J_p^*(p_\star, \theta_\star, \phi_\star) = \sum_{j=0}^{\infty} \sum_{k=-j}^{j} (A_{jk}(p_\star) C_j^k(\theta_\star, \phi_\star) + B_{jk}(p_\star) S_j^k(\theta_\star, \phi_\star)),
\]

(1)