Abstract. Two new families of three-dimensional simple-symmetric periodic orbits are determined numerically in the Sun–Jupiter case of the restricted three-body problem. These families emanate from the ‘vertical-critical’ orbits (\(c_x = 1, c_y = 0\)) of the families i and I of plane symmetric simple-periodic orbits direct around the Sun and the Sun–Jupiter respectively. Further, the numerical technique employed in the determination of these families has been described and interesting results have been pointed out. Also, computer plots of the orbits of these families have been shown in conical projections.

1. Introduction

The planar motion of the restricted three-body problem has been studied exhaustively and an abundance of numerical results has been given while the three-dimensional motion has been neglected and very few results have been given. This is due to the increased complexity of the problem when one more degree of freedom is introduced. Usually the results which have been given for the three-dimensional motion are unrelated pieces of information about families of periodic orbits as there is no clue as to how these families have been generated. The discovery of such families has been based on the ‘continuity’ method of Poincaré (1882), with the exception of some ‘blind’ discoveries of them by using a sophisticated computer program, which converges to a member of such a family, based on the fact that for \(\mu < 0\) the closed motions are as ‘dense’ as they are when \(\mu = 0\). For this purpose a grid search method aimed at locating ‘all’ symmetric periodic motions of the problem of one, two, etc., revolutions has been developed and applied numerically for double-symmetric three-dimensional periodic orbits by Kazantzis and Goudas (1975); the same method applies equally well to other categories of special solutions. The fact that the definition of a family of three-dimensional symmetric periodic orbits requires foremostly the discovery of a starting member has rendered this method very useful. But the question of how these families have been generated has remained. Recent results (Hénon, 1973; Markellos, 1977; Zagouras, 1977; Zagouras and Markellos, 1977; Zagouras and Kalogeropoulos, 1978) give the answer to the question by showing how these families have been generated emanating from members of planar families.

An important effort to compile an ‘atlas’ of three-dimensional symmetric periodic orbits was started in a previous work (Kazantzis 1979, called Paper I hereafter). This
‘atlas’ will give for the first time global information about the three-dimensional symmetric periodic orbits and their properties. For this purpose in Paper I we considered the ‘basic’ families of plane symmetric simple-periodic orbits in the Sun–Jupiter case (μ = 0.000 95) and by studying their stability we revealed the vertically critical members of these families, corresponding to the vertical stability parameter |αv| = 1. These critical members are bifurcations with three-dimensional families of the same multiplicity or twice the multiplicity of the ‘basic’ family. In the present work we consider critical members of the plane ‘basic’ families which are bifurcations with three-dimensional (space) plane-symmetric periodic orbits (or simply, with ‘SPSP’ orbits); the vertical stability parameters of these critical members are αv = 1 and cv = 0. Such members occur on the ‘basic’ families a, c, b, i and l. The families of three-dimensional symmetric periodic orbits which emanate from the families a, c, b have been examined elsewhere (Zagouras and Kazantzis, 1979) because they have common properties. Here we give for the first time the families of ‘SPSP’ orbits which emanate from the planar families i and l. For equations of motion and variation of the problem we refer to Kazantzis (1973) and for any further explanations to Paper I.

2. Determination of ‘SPSP’ Orbits – Stability

The numerical technique employed in the determination of a family of ‘SPSP’ orbits starts from a member of a plane family of symmetric periodic orbits which is marked by the conditions αv = 1 and cv = 0; this member corresponds to a critical point on the characteristic curve of the plane family from which the characteristic curve of a family of ‘SPSP’ orbits emanates and it has the same multiplicity of the plane family (at least at the beginning). We seek the first member of the three-dimensional family by integrating the equations of motion and variation starting with initial state vector

\[ x_0 = (x_{01}, 0, x_{03}, 0, x_{05}, 0), \]

where \( x_{01}, x_{05} \) are the initial components of the critical member of the plane family and \( x_{03} = h \) (\( h \) is a sufficiently small rational positive number for a linear convergence to be operable). We stop the integration when the orbit intersects the \( O x_1 x_3 \) plane for the \( N \)th time, where \( N \) is the multiplicity of the sought periodic orbit. Then, by keeping the initial component \( x_{03} \) constant we seek to adjust approximately corrections (\( \delta x_{01}, \delta x_{05} \)) of the initial state components \( x_{01}, x_{05} \) so that the following conditions of periodicity can be satisfied:

\[ x_2(x_{01} + \delta x_{01}, x_{03}, x_{05} + \delta x_{05}) = 0, \]  
\[ x_4(x_{01} + \delta x_{01}, x_{03}, x_{05} + \delta x_{05}) = 0, \]  
\[ x_6(x_{01} + \delta x_{01}, x_{03}, x_{05} + \delta x_{05}) = 0; \]  

the condition (2) is automatically satisfied at the exact point of intersection of the orbit.