THREE-DIMENSIONAL PERIODIC OSCILLATIONS
GENERATING FROM PLANE PERIODIC ONES AROUND THE
COLLINEAR LAGRANGIAN POINTS

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(Received 3 November, 1978)

Abstract. The three families of three-dimensional periodic oscillations which include the infinitesimal periodic oscillations about the Lagrangian equilibrium points \(L_1, L_2\) and \(L_3\) are computed for the value \(\mu = 0.00095\) (Sun-Jupiter case) of the mass parameter. From the first two vertically critical (\(|a_v| = 1\)) members of the families \(a, b\) and \(c\), six families of periodic orbits in three dimensions are found to bifurcate. These families are presented here together with their stability characteristics. The orbits of the nine families computed are of all types of symmetry \(A, B\) and \(C\). Finally, examples of bifurcations between families of three-dimensional periodic solutions of different type of symmetry are given.

1. Introduction

The infinitesimal periodic oscillations around the collinear Lagrangian points \(L_1, L_2, L_3\) in the restricted three-body problem are continued to finite periodic orbits in the plane of motion of the two primaries as well as in the three dimensions (Moulton, 1920).

In the planar case these finite orbits are grouped into the families \(c, a\) and \(b\) respectively and have been studied numerically by many investigators, e.g., Strömgren (1935); Bartlett (1964); Hénon (1965) for \(\mu = 0.5\); and Markellos (1975) for \(\mu = 0.00095\). Concerning the three-dimensional case, Moulton (1920) has shown that there are three types of finite periodic solutions which are generated from the infinitesimal ones and has approximated a small part of the families associated with two of the three existing types. Bray and Goudas (1967) have computed numerically for \(\mu = 0.4\) the three families corresponding to the first of these three types.

The vertical stability character, on the other hand, of the planar families \(a, b, c\) (Kazantzis, 1979) shows that on each of them there are critical points (\(|a_v| = 1\)) from which other families of three-dimensional periodic orbits (Hénon, 1973) emanate.

In the present paper the possible interrelationships between the families of double symmetry originating from the collinear equilibrium points and the families which are bifurcated from the first two vertical critical orbits of each of the planar families \(a, b\) and \(c\), are investigated.

Astrophysics and Space Science 61 (1979) 389–409. 0004–640X/79/0612–0389 $03.15
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Nine families of three-dimensional periodic orbits are computed for the Sun–Jupiter case \((\mu = 0.00095)\). The linear stability character of each orbit is also examined through the characteristic exponents and many orbit illustrations are given in conical projections.

2. Motion about the Collinear Lagrangian Points \(L_1, L_2, L_3\) in Three Dimensions

In a rotating, barycentric, dimensionless coordinate system with the smaller primary on the positive Ox-axis the differential equations of motion for the circular three-dimensional restricted problem take the form

\[
\dot{x} = f(x),
\]

where

\[
x = (x_1, x_2, x_3, x_4, x_5, x_6) = (x, y, z, \dot{x}, \dot{y}, \dot{z}),
\]

\[
f = (f_1, f_2, f_3, f_4, f_5, f_6) = (x_4, x_5, x_6, 2x_5 + \frac{\partial \Omega}{\partial x_1}, -2x_4 + \frac{\partial \Omega}{\partial x_2}, \frac{\partial \Omega}{\partial x_3})
\]

and

\[
\Omega = \frac{1}{2}(x_1^2 + x_2^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2},
\]

\[
r_1^2 = (x_1 + \mu)^2 + x_2^2 + x_3^2,
\]

\[
r_2^2 = (x_1 + \mu)^2 + x_2^2 + x_3^2.
\]

These equations admit of the Jacobi’s integral

\[
C = 2\Omega - (x_1^2 + x_2^2 + x_3^2).
\]

The five equilibrium points are the real roots of the simultaneous equations

\[
f_i = 0, \quad i = 1, \ldots, 6.
\]

Setting \(x_2 = y = 0\) we can evaluate the position of the collinear Lagrangian points \(L_1, L_2\) and \(L_3\). Here we have designated the point \(L_1\) to lie between the primaries while \(L_2\) rests to the right of Jupiter and \(L_3\) to the left of the Sun. The following table lists the position and the corresponding value of the Jacobi integral in terms of the system adopted in the present paper.

In order to study the motion near the collinear equilibrium points \(l_i = (a_i, b_i, c_i), i = 1, 2, 3\), we consider a perturbation \(\xi^* = (\xi, \eta, \zeta)\). The vectors \(l + \xi^*\) should

<table>
<thead>
<tr>
<th>The collinear Lagrangian points for (\mu = 0.00095)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
</tr>
<tr>
<td>(L_1)</td>
</tr>
<tr>
<td>(L_2)</td>
</tr>
<tr>
<td>(L_3)</td>
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