A UNIFIED THEORY OF POLYNOMIAL EXPANSIONS AND THEIR APPLICATIONS INVOLVING CLEBSCH–GORDAN TYPE LINEARIZATION RELATIONS AND NEUMANN SERIES

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Abstract. Motivated by their potential for applications in several diverse fields of physical, astrophysical, and engineering sciences, this paper aims at presenting a unified study of various classes of polynomial expansions and multiplication theorems associated with the general multivariable hypergeometric function (studied recently by A. W. Niukkanen and H. M. Srivastava), which provides an interesting and useful unification of numerous families of special functions in one and more variables, encountered naturally (and rather frequently) in many physical, quantum chemical, and quantum mechanical situations. Several interesting applications of these general polynomial expansions are considered, not only in the derivations of various Clebsch–Gordan type linearization relations involving products of several Jacobi or Laguerre polynomials, but also to associated Neumann expansions in series of the Bessel functions $J_n(z)$ and $I_n(z)$ (and of their suitable products).

1. Introduction, Notations, and Definitions

It is fairly well-known that hypergeometric series (and, of course, hypergeometric polynomials) in one, two, and more variables occur rather frequently in a wide variety of problems in theoretical physics and applied mathematics (including, for instance, nuclear and neutrino astrophysics), and indeed also in engineering sciences, statistics, and operations research (see, for examples, Srivastava and Karlsson, 1985, Section 1.7, and the various references cited therein). In fact, a considerably vast field of physical, quantum chemical, and quantum mechanical situations (such as Schrödinger’s wave mechanics) lead naturally to various hypergeometric polynomials; for instance, the generalized Bessel polynomials (see, for generalized hypergeometric $pF_q$ notations, Slater, 1966, Chapter 2):

\[
y_n(x, \alpha, \beta) = \sum_{k=0}^{n} \binom{n}{k} \binom{x+n+k-2}{k} k! \left(\frac{x}{\beta}\right)^k = 2F_0 \left[ -n, x+n-1; -\frac{x^2}{\beta} \right],
\]

or, more frequently, the simple Bessel polynomials*

\[
y_n(x) = y_n(x, 2, 2) = 2F_0 \left[ -n, n+1; -\frac{x^2}{2} \right],
\]

* In particular, the polynomials $x^n y_n(x^{-1})$ are the same as the orthogonal polynomials $S_n(x)$ which represent the energy spectral functions for a family of isotropic turbulence fields (see, for details, Srivastava, 1984).

and such classical orthogonal polynomials as the Hermite polynomials

\[ H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k! (n-2k)!} (2x)^{n-2k} = (2x)^n \binom{n}{2} F_0 \left[ \frac{-1/2, -1/2 + 1/2; -1/2}{x^2} \right], \] (3)

the Jacobi polynomials

\[ P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^{n} \binom{\alpha + n}{n-k} \binom{\beta + n}{n-k} \left( \frac{x-1}{2} \right)^k \left( \frac{x+1}{2} \right)^{n-k} = \binom{\alpha + n}{n} \binom{\beta + n}{n} \binom{1}{2} F_1 \left[ -n, \alpha + \beta + n + 1; \frac{1-x}{2} \right], \] (4)

and the Laguerre polynomials

\[ L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{\alpha + n}{n-k} \frac{(-x)^k}{k!} = \binom{\alpha + n}{n} \binom{x+1}{2} F_1 \left[ -n; \alpha + 1; x \right], \] (5)

and indeed also numerous familiar special cases of the Jacobi polynomials including, for example, the Gegenbauer (or ultraspheric) polynomials

\[ C_{n+1/2}^{\nu} (x) = \left( \frac{v+n}{n} \right)^{-1} \binom{2v+n}{n} P_n^{(\nu, \nu)} (x), \] (6)

the Legendre (or spherical) polynomials

\[ P_n(x) = P_n^{(0, 0)}(x), \] (7)

and the Tchebycheff polynomials (of the first and second kinds)

\[ T_n(x) = \left( \frac{n}{n} - \frac{1}{2} \right)^{-1} P_n^{(-1/2, -1/2)}(x) \] (8)

and

\[ U_n(x) = \frac{1}{2} \left( \frac{n+1}{n+1} \right)^{-1} P_n^{(1/2, 1/2)}(x). \] (9)

In view of the well-known relationships

\[ y_n(x, \alpha, \beta) = n! \left( -\frac{x}{\beta} \right)^n L_n^{(1-\alpha-2n)} \left( \frac{\beta}{x} \right) \] (10)

and

\[ H_{2n+\delta}(x) = (-1)^n 2^{2n+\delta} n! x^{\delta} L_n^{(\delta-1/2)}(x^2) \quad (\delta = 0 \text{ or } 1), \] (11)

all of the above-mentioned orthogonal polynomials (and many more) are easily recoverable from the classical Jacobi and Laguerre polynomials.