EXACT SOLUTION OF TRANSPORT EQUATION IN
FINITE MULTIPLYING MEDIA

S. R. DAS GUPTA and S. K. BISHNU
Department of Mathematics, University of North Bengal, Darjeeling, West Bengal, India

(Received 30 May, 1980)

Abstract. One speed neutron transport equation in a finite multiplying medium is exactly and uniquely solved following the first author's new technique (Das Gupta, 1978b) based on Laplace transformation and Wiener–Hopf technique, to obtain the angular neutron density at any depth. Criticality condition is derived.

Exact solution of mono-energetic neutron transport equation in a finite uniform, plane-parallel isotropically scattering multiplying slab having supply of neutrons only through fission, is obtained in a simple form following the author's new technique (cf. Das Gupta, 1978b). The equation has been attacked by several authors following different methods (cf. Kobayashi et al., 1968). The transport equation appropriate to the problem is (cf. Case and Zweifel, 1967, Equation (2a), p. 152)

\[ \mu \frac{d\psi(t, \mu)}{dt} = \psi(t, \mu) - \omega \psi_0(t), \quad \omega > 1, \]  

\( \omega = (\gamma_s + c\gamma_a)/\gamma, \) \( \gamma_s \) and \( \gamma_a \) are the macroscopic cross-sections for scattering and absorption respectively. \( \gamma = \gamma_s + \gamma_a, c \) is the multiplication factor. \( c > 1 \Rightarrow \omega > 1. \) \( c < 1 \Rightarrow \omega < 1. \) The angular flux \( \psi(t, \mu) \) in the direction \( \cos^{-1} \mu \) and at the depth \( t \) measured in units \( \gamma, 0 \leq t \leq b, \) has a symmetry as given by Equation (2, i) and satisfies the boundary condition (2, ii) in absence of any beam incident on the slab surfaces of the form

(i) \[ \psi(t, \mu) = \psi(b - t, -\mu), \]  

(ii) \[ \psi(0, -\mu) = \psi(b, \mu) = 0, \] \( 0 < \mu \leq 1, \)

and

\[ \psi_0(t) = \frac{1}{2} \int_{-1}^{1} \psi(t, \mu) \, d\mu, \]  

and

is the scalar neutron flux. With \( f(t) = 0 \) on \([0, b]^t\) we define the Laplace transform of \( f(t) \) by

\[ f^*(s) = s \int_{0}^{\infty} f(t) e^{-st} \, dt, \quad \text{Re } s > 0, \]  

\[ = s \int_{0}^{b} f(t) e^{-st} \, dt \]  

regular in the \( s \)-plane as \( b < \infty, \) \( f(t) \) is then given by

\[ f(t) = \frac{1}{2\pi i} \int_{-i\omega}^{i\omega+\delta} e^{st} f^*(s) \, ds/s, \quad 0 < \delta < 1. \]  

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Taking $\psi(t, \mu)$ and $\psi_0(t)$ to vanish on $[0, b]^2$, the formal solution of Equation (1) is

$$M(\mu) = \psi(0, \mu) - e^{-b\mu}\psi(b, \mu) = \omega\psi_0^+(1/\mu), \text{ regular for } |\mu| > 0.$$  \hfill (4)

If we put

$$N(\mu) = e^{-b\mu}M(-\mu) = e^{-b\mu}\psi(0, -\mu) - \psi(b, -\mu), \hfill (4a)$$

from (4) we have by the boundary condition Equation (2)

$$M(\mu) = \psi(0, x) \text{ on } [0, 1],$$

$$N(\mu) = -\psi(b, -x) = -\psi(0, x) \text{ on } [0, 1]. \hfill (5)$$

Subjecting Equation (1) to Laplace transformation as defined by Equation (3a) we get

$$\psi^*(s) = \frac{\mu s \left[ \psi(0, \mu) - e^{-b\mu}\psi(b, \mu) \right] - \omega\psi_0^+(s)}{\mu s - 1}. \hfill (6)$$

Operating on Equation (6) with $\frac{1}{\mu} \int_0^1 d\mu$ we get Equation (7a), and eliminating $\psi_0^+(s)$ from Equations (7a) and (6) we get successively (see Equations (4) and (5))

$$\omega\psi_0^+(s) = f^+(s) - f^+_0(s), \hfill (7a)$$

$$\psi^*(s, \mu) = F^+(s) - F^+_0(s), \hfill (7b)$$

where we put

$$f^+(s) = \frac{G^+(1/s)}{T(1/s)}, \hfill (8a)$$

$$f^+_0(s) = \frac{e^{-b}sG^-(1/s)}{T(1/s)}, \hfill (8b)$$

$$F^+(s) = \frac{\mu s \psi(0, \mu) - G^+(1/s)/T(1/s)}{\mu s - 1}, \hfill (9a)$$

$$F^+_0(s) = \frac{e^{-b} \mu s \psi(b, \mu) - G^-(1/s)/T(1/s)}{\mu s - 1}. \hfill (9b)$$

Eliminating $\psi_0^+(s)$ from Equations (4) and (7a) we get (on setting $s = 1/z$) the integral equation

$$T(z)M(z) = G^+(z) - e^{-b/z}G^-(z), \hfill (10)$$

where we write

$$G^+(z) = \frac{\omega}{2} \int_0^1 \frac{xM(x) \, dx}{x - z}, \hfill (11)$$

and

$$\psi_0^+(1/\mu) \rightarrow O(1/z) \text{ as } z \rightarrow \infty, \text{ and is regular on } [0, 1]^c.$$  \hfill (11a)