THE MODIFIED LAGRANGIAN EQUATIONS OF
GENERAL-RELATIVISTIC HYDRODYNAMICS FOR A
NONIDEAL FLUID

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Abstract. The general-relativistic equations of hydrodynamics for a nonideal fluid are derived in the modified Lagrangian form. Together with the zeroth and first moment equations of radiative transfer derived by Morita and Kaneko (1986), the equations provide a complete set of the modified Lagrangian equations of radiation hydrodynamics. The equations of hydrodynamics are specialized for a thermally conducting, Newtonian viscous fluid in the modified Lagrangian form, which are the generalization of the special-relativistic equations of hydrodynamics derived by Greenberg (1975).

1. Introduction

The interaction of radiation with matter plays an essential role in studying the nuclei of Seyfert galaxies and quasars. In such problems, radiative energy density and pressure are comparable to or overwhelm those of the fluid, and the fluid velocity becomes relativistic near black holes. Also, the effects of viscosity may be of significant importance in determining the dynamics and structure of accretion disks. Therefore, the equations of radiation hydrodynamics applicable to the study of active galactic nuclei must take into account (1) the fluid velocity accurate to terms of all orders in \((v/c)\), (2) the curvature in spacetime, and (3) material and radiative viscosity.

With these facts in mind, we have recently developed the method of deriving the comoving-frame equation of radiative transfer and its moment equations in the modified Lagrangian form, in which the physical quantities are evaluated in the comoving frame and the spacetime variables and the fluid velocity are measured in the inertial frame (Morita and Kaneko, 1986). The essence of our approach is that two sets of orthonormal bases fixed in the inertial and comoving frames exactly follow a local Lorentz transformation. This enables us to deal with terms of all orders in \(v/c\) in flat or curved spacetime and to write the comoving-frame equations in terms of the inertial-frame spacetime variables and the ordinary 3-velocity of the fluid. For various descriptions of the equation of radiative transfer, we refer the readers to the monographs by Pomraning (1973) and by Mihalas and Mihalas (1984) and the reviews by Munier and Weaver (1986a, b).

The modified Lagrangian description we adopted for the equation of radiative transfer may also be appropriate to general-relativistic hydrodynamics, because this description is identical to those used in the post-Newtonian (Chandrasekhar, 1965, 1969) and special-relativistic hydrodynamics (Landau and Lifshitz, 1958). Although such a nomenclature is not used, Greenberg (1971, 1975) have derived the modified Lagrangian
equations of hydrodynamics for a nonideal fluid both in the post-Newtonian approximation and in special relativity. In this paper, we derive the general-relativistic equations of hydrodynamics for a nonideal fluid in the modified Lagrangian form. Together with our zeroth- and first-moment equations of radiative transfer (Morita and Kaneko, 1986), they provide a complete set of the equations of radiation hydrodynamics.

In Section 2, we first review the general-relativistic equations of hydrodynamics for a nonideal fluid in the frame-independent form (Misner et al., 1973), and then, we write them in an orthonormal coordinate system fixed in the comoving frame. In Section 3, following Morita and Kaneko (1986), we write the kinematic quantities of the fluid in terms of the inertial-frame spacetime variables and the ordinary 3-velocity of the fluid. The modified Lagrangian equations of hydrodynamics accurate to terms of all orders in v/c and valid in flat or curved spacetime are given in Section 4. The non-zero components of the kinematic quantities of the fluid are also given in the Appendix.

2. The General-Relativistic Equations of Hydrodynamics for a Nonideal Fluid in the Comoving Frame

2.1. Frame-independent Equations of Hydrodynamics

The general-relativistic equations of hydrodynamics are described by (1) the law of conservation of baryon number and (2) the law of energy-momentum conservation.

The law of conservation of baryon number is expressed by

$$\nabla \cdot (n\mathbf{u}) = 0,$$

where $n$ and $\mathbf{u}$ are the baryon number density measured in the comoving fluid frame and the 4-velocity of the fluid ($\mathbf{u} \cdot \mathbf{u} = -1$), respectively; $\nabla$ is the covariant derivative, or the connection, and the upper figure 4 on the 3-dimensional operator $\nabla$ denotes the 4-dimensional one (Morita and Kaneko, 1986).

The law of energy-momentum conservation for a matter is given by

$$\nabla \cdot T_m = Q,$$

where $Q$ is the 4-force density acting on the matter. The energy-momentum tensor for a nonideal fluid matter is written as

$$T_m = \rho \mathbf{u} \otimes \mathbf{u} + q \otimes \mathbf{u} + \mathbf{u} \otimes q + t$$

with

$$t = P \cdot T_m \cdot P = p P + \Pi,$$

where $\rho = \mathbf{u} \cdot T_m \cdot \mathbf{u}$ is the mass-energy density of the fluid, $q = - \mathbf{u} \cdot T_m \cdot P = - P \cdot T_m \cdot \mathbf{u}$ is the energy-flux 4-vector, $p$ is the isotropic pressure of the fluid, and $\Pi$ is the viscous tensor. The projection tensor $P$ is expressed in terms of the 4-velocity $\mathbf{u}$ and the metric tensor $g$ as

$$P = \mathbf{u} \otimes \mathbf{u} + g,$$