Abstract. The integrals $J_{m,\gamma}$ were introduced by Kopal for the numerical evaluation of the light changes exhibited by eclipsing binaries when both the tidal and rotational distortions are taken into account.

This paper is a sequel to a previous one to appear in this journal and aims at ascertaining some recursion formulae for these integrals to alleviate the computational complexity of the problem.

Using a relationship existing between the $J$-integrals and the Appell hypergeometric series of the first kind, we have been able to obtain recursion formulae affecting all three parameters $m, \beta, \gamma$ of these integrals. The present stage of development has also allowed for a complete enumeration of all independent recursion formulae applicable to the case of partial eclipses.

Various recursion formulae, given here for the first time, generalize previous results by Kopal which were valid for $m=0$ or $\gamma=0$.

1. Introduction

The motivation for this paper is the numerical evaluation of the light changes exhibited by a binary system of stars. We are assuming here that the components of the system are subject to both tidal and rotational deformations, furthermore that the inclination of their orbital plane to the line of sight gives rise to eclipses. In the fundamental work on this subject, Kopal (1959, pp. 188–207), the loss of light for a distorted eclipsing system has been expressed in terms of a new class of functions and integrals, respectively labelled alpha-functions and $J$-integrals. The expeditious numerical evaluation of these light changes depends essentially on the recursion formulae for the $J$-integrals which hold for integral increments of their parameters, i.e., for contiguous values of their parameters. Some recursion formulae valid for particular values $m$ and $\gamma$ have already been given in Kopal (1959, pp. 215–216).

In a previous paper (Lanzano, 1976), henceforth referred to as paper I, some new and more comprehensive recursion formulae for the $J$-integrals were obtained by making use of the representation of the $J$-integrals in terms of the Appell hypergeometric series of the first kind. Let us recall at this stage that the definition of $J$-integrals varies according as to whether one is dealing with the numerical evaluation of a partial or an annular eclipse. The purpose of this paper is to complement the results of I by providing a deeper understanding of the problem and also the enumeration of all recursion formulae for the $J$-integrals of a partial eclipse. In particular, recursion formulae will be exhibited that entail the variation of all three parameters in question.

For the purpose of rendering the present paper self sufficient, we shall recall here the
essential definitions and relationships which already have been shown in I.

The J-integral for a partial eclipse is defined as follows, see Kopal (1959), p. 213:

$$J^m_{\mu, \gamma} = \frac{(2\kappa)^{\beta + \gamma + 2}}{2\pi} \int_0^1 u^{\beta/2}(1 - u)^{\gamma/2}(1 - \kappa^2 u)^{\beta/2}(1 - 2\kappa^2 u)^m \, du,$$

where \( \beta \) and \( \gamma \) are arbitrary parameters subject to the condition \( \beta + \gamma + 2 > 0 \),

while \( m \) is zero or a positive integer; the modulus \( \kappa \) is given by

$$2\kappa^2 = 1 - \mu,$$

where \( \mu \) is a function of the two radii of the stars and of the separation of their centers.

The fundamental relationship expressing the J-integral in terms of the gamma-function and of the Appell hypergeometric series of the first kind is

$$\frac{\Gamma(A)\Gamma(C - A)}{\Gamma(C)} F^{(1)}(A; B, B'; C; X, Y) = \int_0^1 U(u) \, du,$$

where

$$U(u) = u^{A-1}(1 - u)^{C-A-1}(1 - uX)^{-B}(1 - uY)^{-B'},$$

and

$$F^{(1)}(A; B, B'; C; X, Y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(A)_r(B)_r(B')_s}{s! (C)_r} X^r Y^s$$

is the Appell hypergeometric series of the first kind, where we have denoted

$$(A)_r = A(A + 1) \cdots (A + r - 1), \quad (A)_0 = 1.$$  

This series is convergent for \( |X| < 1, |Y| < 1 \) and is a generalization of Gauss hypergeometric series

$$F(A; B; C; X) = \sum_{r=0}^{\infty} \frac{(A)_r(B)_r}{r! (C)_r} X^r,$$

to which it reduces when \( Y = 0 \) or \( B' = 0 \).

Upon using Equations (3) and (4), the expression for the J-integral becomes

$$J^m_{\mu, \gamma} = \frac{(2\kappa)^{\beta + \gamma + 2}}{2\pi} \frac{\Gamma(1 + \beta/2)\Gamma(1 + \gamma/2)}{\Gamma(2 + \beta/2 + \gamma/2)} \times$$

$$\times F^{(1)} \left( 1 + \frac{\beta}{2}; -\frac{\beta}{2}, -m; 2 + \frac{\beta}{2} + \frac{\gamma}{2}; \kappa^2, 2\kappa^2 \right).$$

We remark in passing that \( J^0_{\mu, \gamma} \) is related to the Gauss hypergeometric series.