Abstract. In this paper we shall investigate the energy of close binary systems of constant momentum taking into consideration the first-order effects of rotation and tidal attraction of the components of finite size. The equations for the momentum and the energy of the system will be set up in Section 2, making use of terms including the effects of finite size of the components of finite degree of central condensation. In Section 3 perturbation theory is applied to these equations using the results of Kopal (1972b) as our initial values. In Section 4 we shall compare our results with the initial values and then discuss variations in our constants and the application to various real systems.

1. Introduction

In the study of close binary systems we not only consider the fact that the radii of the components are comparable with the separation, but the effects due to tidal and rotational distortion from the spherical configuration are also observable. Chief among the observable effects is that of the rotation of the apsides of the orbit.

In the paper by Kopal (1972b) (which we shall henceforth call Paper I) the components of the various motions for the minimum energy were derived and the equations for the rate of evolution expressed. This paper considered that for the equations of energy and momentum to be soluble in closed form the terms due to tidal and rotational effects were ignorable. It is the purpose of this paper to take into account the rotational and tidal effects and the effects of inequality of the masses, by beginning with Kopal's results and performing a perturbation calculation on the equations.

The methods described by Messiah (1961) for handling perturbation equations will be used in our work. This is justified in our being able to identify the equations as those containing linear operators of the form

\[ H_{op} \psi = E. \]  

We shall investigate the effects of the mass distribution on the solutions of our equations. It is to be expected that if the masses are unequal we may no longer have synchronous rotation; and this may bring our solutions more closely in line with observations. In investigating the tidal effects we shall investigate the way in which the apsidal motion constant varies and in that way the effect of the polytrope on the motion. Using this we may reconsider the evolutionary state of systems on the Main Sequence and those evolved away from the Main Sequence, such as Algol.

2. Equations of Energy and Momentum

In Paper I the equations of energy and momentum were formulated in their entirety.
Assuming small values of eccentricity, the momentum was given by

\[ H = H_0 + C_1 \omega_1 + C_2 \omega_2 + (\tilde{C}_1 + \tilde{C}_2) \Omega, \]  
\[ \text{(2.1)} \]

where

\[ H_0 = G^{2/3} m_1 m_2 (m_1 + m_2)^{-1/3} \Omega^{-1/3}, \]  
\[ \text{(2.2)} \]

\[ C_i = m_i \tilde{S}_i^2 + \frac{2 (k_2)_i \omega_i R_i^5}{9G} + \ldots, \]  
\[ \text{(2.3)} \]

\[ \tilde{C}_i = \frac{1}{3} m_{3-i} (k_2)_i R_i^5 + \ldots. \]  
\[ \text{(2.4)} \]

Reducing the above equation to a non-dimensional form, we have

\[ h = x + \frac{\dot{\lambda}}{x^9} + \kappa_1 y + \kappa_2 z + \kappa'_1 y^3 + \kappa'_2 z^3, \]  
\[ \text{(2.5)} \]

where

\[ \kappa_i = m_i \tilde{S}_i^2 / [(m_1 + m_2) L^2], \]  
\[ \text{(2.6)} \]

\[ \kappa'_i = \frac{5}{6} (k_2)_i R_i^5 (m_1 m_2)^{3/2} / [(m_1 + m_2)^3 L^5], \]  
\[ \text{(2.7)} \]

\[ \lambda = \frac{1}{3} (m_1 m_2)^{3/2} L^5 \left[ m_2 (k_2)_1 R_1^5 + m_1 (k_2)_2 R_2^5 \right], \]  
\[ \text{(2.8)}^* \]

where the \( C_i \)'s and the \( \tilde{C}_i \)'s have been averaged over the period of the revolution. This is an augmented form of equation (2.21) of Paper I.

From Equation (2.36) of Paper I, the total energy is

\[ E = \frac{1}{2} \left\{ C_1 \omega_1^2 + C_2 \omega_2^2 + (\tilde{C}_1 + \tilde{C}_2) \Omega^2 \right\} + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} A^2 \Omega^2 - \frac{G m_1 m_2}{A} \left\{ 1 + \delta_{12} \right\}, \]  
\[ \text{(2.9)} \]

where

\[ 2S_{12} = \sum_{i=1}^{2} \sum_{j=2}^{4} (k_j)_i m_{3-i} \left( \frac{R_i}{m_i} \right)^{2j+1} - \sum_{i=1}^{2} \frac{\omega_i}{2\pi G \tilde{G}} (k_2)_i \left( \frac{R_i}{r} \right)^2, \]  
\[ \text{(2.10)} \]

Now if we neglect terms containing \( k_3 \) and \( k_4 \), which are orders of magnitude smaller than \( k_2 \), and convert again to dimensionless unit, the energy becomes

\[ 2E = -\frac{1}{x^2} + \kappa_1 y^2 + \kappa_2 z^2 + \frac{\lambda - \epsilon}{x^4} + \kappa'_1 y^4 + \kappa'_2 z^4 + \frac{\eta_1 y^2 + \eta_2 z^2}{x^6}, \]  
\[ \text{(2.11)} \]

where we again have averaged over the period. The expressions for \( \kappa_i, \kappa'_i \) and \( \lambda \) are as given before and

\[ \epsilon = \frac{1}{2} \left[ \frac{(m_1 m_2)^{5/2}}{(m_1 + m_2)^{5/2}} \right] \left\{ (k_2)_1 R_1^5 m_2 + (k_2)_2 R_2^5 m_1 \right\} / L^5, \]  
\[ \text{(2.12)} \]

\[ \eta = \frac{3 m_i^{3/2} m_{3-i}^{5/2}}{(m_1 + m_2)^{3/2}} (k_2)_i R_i^5 / L^5. \]  
\[ \text{(2.13)} \]

* We note that our expression for \( \lambda \) differs from that of Paper I by the factor of \( A^9 \). (Alexander, 1972).