EXACT TWO-SOLITON SOLUTIONS TO THE EINSTEIN GRAVITATIONAL FIELD EQUATIONS

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Abstract. Following our previous work on the existence of 1-soliton solution to the Einstein gravitational field equations in the presence of a spherically-symmetric static background field, we have found six sets of analytical 2-soliton solutions to the Einstein field equations under a certain ansatz in the absence of the stated background field. Numerical analysis shows that if the two solitons of the transverse nature are injected at space variable $z \to \pm \infty$, the longitudinal field component $g_{33}$ will acquire non-zero values for a bounded spatial region at later time. The nature of the solitons becomes rather complex when they interact. The amplitude $g_{\mu \nu}$ of each soliton may change its magnitude resulting from the interaction. We have found that we might interpret the evolution of one field component as the gravitational instanton in our solutions. We must remark also that the total energy of the interacting solitons remains constant, as expected, at all time. These solutions correspond to the situation where the Riemann tensor is in general non-zero and are truly non-trivial solutions.

1. Introduction

Einstein already found plane wave solution to his gravitational field equation $R_{\mu \nu} = 0$ in 1906 under the weak field approximation (cf. Misner et al., 1973). However, it is a difficult task to build up equipment sensitive enough to detect gravity waves experimentally if the waves are plane waves. We are still searching for wave solutions to this equation in the hope of obtaining wave solutions of the type(s) other than plane wave ones. Such solutions, if so found, would give us more information as to how gravity waves could be detected.

Even though in the simple case where the gravity waves are propagating in a vacuum the gravitational field equation $R_{\mu \nu} = 0$ is already highly nonlinear. Based on nonlinear physics study developed in the last two decades, we know that nonlinear equations often contain soliton solutions. It is thus natural to investigate whether the field equation contains soliton solutions also. If soliton solutions do exist and are physical, the possibility of detecting gravity waves would be increased tremendously and we can understand more about the space-time structure of our Universe.

By use of the inverse scattering method, Belinskii and Zakharov (1978) obtained solutions which show soliton-like composition under certain limiting conditions and approximations. Following the above work, Belinskii and Fargion (1980) obtained another set of soliton-like solutions which do not contain the usual propagating factor; under certain limiting conditions, the speed of the soliton-like disturbance is greater than the speed of light in free space. We consider that these solutions may not be physical. Later, Ib\'a\'nez and Verdaguer (1983) obtained a set of 4-soliton solutions in a limited space region using again the inverse scattering method.

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More recently, starting from the Bondi metric and using the general differential geometry methodology, one of us obtained a set of unidirectional 1-soliton solution to $R_{\mu\nu} = 0$ (Tao, 1987). In that investigation, it was shown that the solutions are non-singular and the energy of the disturbance is positive definite. We then included a spherically-symmetrical gravitational background field in the Einstein gravitational field (cf. Fung and Tao, 1987) as we consider such a field to be more realistic in the sense that any ground base observation is inevitably carried out in the terrestrial gravitational field. We have found ‘composite soliton’ solutions (namely, disturbance which is made up of more than one peak) in our investigation; the soliton structure is very obvious. It should also be remarked that in the presence of the terrestrial field, the longitudinal component of the 'mixed-mode soliton' could not be zero. In other words, longitudinal type of 1-soliton disturbance exists in the propagating gravitational waves in the presence of the stated background field.

In this study, we generalize the previous work on unidirectional 1-soliton solution to $R_{\mu\nu} = 0$ by looking for 2-soliton solutions using the same differential geometry approach. These solutions are nontrivial ones, because they correspond to nonzero Riemann tensor in the space-time domain. In our analysis, the properties of collisions between two solitons are demonstrated with numerical examples.

2. Exact Two-Soliton Solutions to the Einstein Gravitational Field Equation in Vacuum

The gravitational field is described by the diagonal metric

$$\begin{align*}
\mathrm{d}s^2 &= -g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \\
&= A(z, t) (c^2 \mathrm{d}t^2 - \mathrm{d}z^2) - B(z, t) \, \mathrm{d}x^2 - C(z, t) \, \mathrm{d}y^2.
\end{align*}$$

(1)

The Christoffel symbol $\Gamma^\alpha_{\mu\nu}$ is related to the metric tensor by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}).$$

(2)

By use of (1), the non-zero components of the Christoffel symbol are

$$\begin{align*}
\Gamma^1_{13} &= \frac{1}{2B} \, B'_3, & \Gamma^3_{22} &= -\frac{1}{2A} \, C'_3, & \Gamma^0_{22} &= \frac{1}{2A} \, C_0, \\
\Gamma^1_{10} &= \frac{1}{2B} \, B'_0, & \Gamma^3_{33} &= \frac{1}{2A} \, A'_3, & \Gamma^0_{33} &= \frac{1}{2A} \, A_0', \\
\Gamma^2_{23} &= \frac{1}{2C} \, C'_3, & \Gamma^1_{30} &= \frac{1}{2A} \, A'_0, & \Gamma^0_{30} &= \frac{1}{2A} \, A'_3, \\
\Gamma^2_{20} &= \frac{1}{2C} \, C'_0, & \Gamma^3_{00} &= \frac{1}{2A} \, A'_3, & \Gamma^0_{00} &= \frac{1}{2A} \, A'_0, \\
\Gamma^3_{11} &= -\frac{1}{2A} \, B'_3, & \Gamma^0_{11} &= \frac{1}{2A} \, B'_0.
\end{align*}$$

(3)