FREEZING-IN CONDITION FOR A MAGNETIC FIELD
AND CURRENT SHEETS IN PLASMA*

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Abstract. In the ideal magnetohydrodynamic approximation it is shown that for physically permissible boundary conditions there may exist some lines on which the freezing-in condition is not valid. Such singular lines are closed magnetic lines of force and lines with both ends on the boundary surface. By analogy with the singular lines of a potential magnetic field the conclusion is made that X-type singular lines are the place where current sheets (sheet pinches) appear in plasma, whereas on O-type singular lines quasi-cylindrical pinches of a usual type appear.

In magnetohydrodynamics of a well-conducting medium the known condition of a frozen-in magnetic field is widely used. Alfvén was one of the first to realize and to use this condition (Alfvén, 1950). He was also one of the first to express doubts about the possibility of automatically applying this condition to low-density cosmic plasma (Alfvén and Fälthammar, 1963; Alfvén, 1977). These doubts and the corresponding arguments were based on the consideration of specific kinetic effects in low-density plasma, leading to the appearance of electric fields directed along the magnetic field. As is well known (see also below), the frozen-in condition excludes such fields.

The object of the present paper is to discuss the possibility that longitudinal electric fields may already appear within a purely hydrodynamic approach, irrespective of plasma-kinetic effects. In other words, it will be shown that a violation of the freezing-in condition is possible within the framework of ideal magnetohydrodynamics. More precisely, we shall show that in a perfectly conducting medium there may exist some regions (singular lines) near which the solution of ideal magnetohydrodynamic equations has jump-like singularities of the magnetic field and, correspondingly, infinite-density currents. These singularities vanish at an arbitrarily high but finite conductivity.

Thus, the condition of an arbitrarily high conductivity does not necessarily lead to the limit of infinite conductivity. Correspondingly, the frozen-in condition is not a straightforward consequence of high conductivity, but also implies the absence of the singular lines mentioned above.

Let us present the usual derivation of the frozen-in condition. This equation is a consequence of the Maxwell induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = -c \text{rot} \mathbf{E},$$

(1)

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and of the Ohm's law for moving conductor

\[ j = \sigma \left( E + \frac{1}{c} V \times B \right), \]

which in the infinite conductivity limit can be written in the form

\[ E = -\frac{1}{c} V \times B. \]

Combining (1) and (3) we get the following magnetohydrodynamic equation of induction (freezing-in equation)

\[ \frac{\partial B}{\partial t} = \text{rot} [V \times B]. \]

From this equation it follows that the magnetic field flux is conserved through any surface \( S \) which moves along with the plasma (local velocity \( V(r, t) \)). Indeed, the change of flux of the arbitrary vector field \( B \) through the moving surface can be expressed as

\[ \frac{d}{dt} \int_S B \, dS = \int_S \left( \frac{\partial B}{\partial t} - \text{rot} [V \times B] + V \text{div} B \right) dS. \]

In the case of a magnetic field satisfying Equation (9) and the condition \( \text{div} B = 0 \) we have (Elsässer, 1947)

\[ \frac{d}{dt} \int_S B \, dS = 0, \]

where \( S \) is an arbitrary substantial surface, i.e. the one moving along with the particles of the medium. It follows immediately that magnetic lines of force are frozen-in. So, substituting a thin force tube for a magnetic line of force we see that in the course of plasma motion, those particles remain in the tube. More formally, a magnetic line of force can be considered as an intersection of two surfaces with a zero magnetic flux through them. Then, by virtue of Equation (6), an intersection of such substantial surfaces will always remain a magnetic line of force and will evidently cross the same plasma particles.

For the given velocity field \( V(r, t) \), Equation (4) determines unambiguously the time evolution of the magnetic field \( B(r, t) \). Thus, for an incompressible liquid Equation (4) has the solution (see, e.g., Syrovatskii, 1957)

\[ B(r, t) = B_0(r_0) + (B_0(r_0) \nabla \xi(r_0, t)), \]

where \( \xi(r_0, t) \) is the displacement of the medium from the initial position \( r_0 \), where the magnetic field was equal to \( B_0(r_0) \) and \( r = r_0 + \xi(r_0, t) \).

Thus, for any continuous velocity field the freezing-in equation (Equation (4)) determines the unique evolution of the initial magnetic field. Such a formulation of the problem is used, as a rule, in the so-called kinematic approach in magneto-