ON SECULAR STABILITY OF TIDALLY DISTORTED
STARS OF ARBITRARY STRUCTURE

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(Received 3 October, 1973)

Abstract. The aim of the present paper will be to develop a theory which should make it possible to investigate secular stability of close binary systems, consisting of tidally-distorted components of arbitrary internal structure, by a minimization of the potential energy of the system as a whole. In the second section which follows brief introductory remarks, appropriate expressions for the total potential energy of a close binary will be formulated. Section 3 will be concerned mainly with the nature of the tide-generating potential, and its effects on the shape of each star. In Section 4, the amplitudes of partial tides raised by this potential will be specified, for stars of arbitrary structure, correctly to terms of second order in superficial distortion; and in Section 5 we shall investigate the effects of interaction between rotation and tides to the same degree of approximation. The concluding Section 6 will then contain an explicit formulation of different constituents adding up to the total potential energy of the system, which can be used as a basis for its secular stability by the methods outlined already in our previous investigation (Kopal, 1973).

1. Introduction

In a preceding paper which recently appeared in this journal (Kopal, 1973; hereafter referred to as Paper I), we investigated the secular stability of rapidly rotating stars of arbitrary structure by a minimization of their total potential energy, by methods which go back to Dirichlet and Poincaré. In the present paper we wish to extend the scope of our investigation to an inquiry as to the secular stability of double-star systems, which consist of two components of arbitrary structure distorted by mutual tidal action. Such an investigation will require a broadening of our definition of the potential energy of the binary configuration as a whole; for the total potential energy of a close binary consists not only of the potential energies of the constituent stars; as their sum must be augmented by the potential energy of their orbit - arising from the fact that the two configurations are not infinitely far apart.

In Section 2 which follows this introduction, an appropriate expression for the total potential energy of a close binary system will be formulated, and expressed in terms of the mass-integrals of the respective potential functions. A determination of these potential functions for tidally-distorted configurations proves, however (see, e.g., Kopal, 1960), to be much more involved than the treatment of the rotational distortion; for the tide-generating potential contains many more harmonic terms in its expansion than an expression for the centrifugal force. As a result, we shall be unable to advance in the present paper our knowledge of the expressions for the respective potential function due to tidal distortion beyond the second-order terms previously investigated by the writer (Kopal, 1960; Chapter III); and the effects of the interaction
between rotation and tides will likewise be limited to those arising from the products of the respective first-order terms (Kopal, 1960; Chapter IV).

2. Potential Energy

Let – in conformity with the notations used in Paper I – \( W \) denote the total potential energy of our binary system; and \( \Omega_{1,2} \), the potential functions of the constituent components. If, moreover, \( d m_{1,2} \) denote the mass elements of the respective stars, then

\[
-W = \frac{1}{2} \int (\Omega_1 + \Omega_2) (d m_1 + d m_2)
= \frac{1}{2} \int \Omega_1 \ d m_1 + \frac{1}{2} \int \Omega_2 \ d m_2 + \frac{1}{2} \int \Omega_1 \ d m_2 + \frac{1}{2} \int \Omega_2 \ d m_1,
\]

where

\[
W_i = -\frac{1}{2} \int \Omega_i \ d m_i, \quad i = 1, 2, \tag{2.2}
\]

denote the potential energy of each component; and

\[
W_{12} = -\frac{1}{2} \left\{ \int \Omega_1 \ d m_2 + \int \Omega_2 \ d m_1 \right\}, \tag{2.3}
\]

the potential energy arising from interaction of the two stars. It can be shown (cf., e.g., MacMillan, 1958) that if the potentials \( \Omega_{1,2} \) are continuous everywhere (though their derivatives may be discontinuous over the boundary surface of each star), satisfy the Laplace equation \( \nabla^2 \Omega_{1,2} = 0 \) on the exterior, and vanish at infinity, the two integrals on the r.h.s. of Equation (2.3) are equal; and their sum represents the potential energy \( W_{12} \) arising from the fact that the two components are situated at a finite distance from each other.

In order to obtain more explicit expressions for the respective mass-integrals of \( \Omega_i \), advantage can be taken of the fact that, in the interior of each configuration, \( \Omega_i \) is bound to satisfy Poisson's equation

\[
\nabla^2 \Omega_i + 4\pi G \rho_i = 0, \tag{2.4}
\]

where \( \rho_i \) denotes the internal density of the respective configuration; and \( G \), the gravitation constant. If we multiply now the foregoing equation by \( \Omega_i \) and integrate with respect to the volume element \( d \tau_i \) related with the mass element \( d m_i \) by

\[
d m_i = \rho_i \ d \tau_i, \tag{2.5}
\]

we easily find that

\[
\int \Omega_i \ d m_i = -\frac{1}{4\pi G} \int \Omega_i \nabla^2 \Omega_i \ d \tau_i, \tag{2.6}
\]

the limits of which can temporarily be regarded as arbitrary.