SOME VACUUM SOLUTIONS IN THE 
SCALE-COVARIANT THEORY OF THE UNIVERSE

(Letter to the Editor)

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Abstract. Static Friedmann-Robertson-Walker vacuum models are derived in the scale-covariant theory. Specific functional forms are obtained for the gauge function which occurs in the theory. This is in contrast to the nonstatic vacuum solutions where the gauge function is arbitrary.

1. Introduction

This article is a continuation of our previous one (Beesham, 1986) in which we derived the nonstatic vacuum Friedmann-Robertson-Walker models in the scale-covariant theory. Here we derive the static vacuum models. Our reasons are twofold; firstly, for the sake of completeness and, secondly, to show that they lead to specific forms for the gauge function which arises in the theory.

The field equations in the scale-covariant theory (Canuto et al., 1977) are

\[ R_{ab} - \frac{1}{2} R g_{ab} + f_{ab}(\beta) + \Lambda g_{ab} = G T_{ab}, \]

\[ \beta^2 f_{ab}(\beta) = 2 \beta \beta_{a;b} - 4 \beta_a \beta_b - (2 \beta \beta_c - \beta_c \beta^c) g_{ab}, \]

where \( \beta \) is a scalar function (the gauge function) satisfying \( 0 < \beta < \infty \). In these equations, \( R_{ab} \) is the Ricci tensor; \( R \), the Ricci scalar; \( g_{ab} \), the metric tensor; \( \Lambda \), the cosmological 'constant'; \( G \), the gravitational 'constant'; and \( T_{ab} \), the energy momentum tensor. We are using units in which the speed of light is unity and the constant \( 8 \pi \) has been absorbed into \( G \). \( \Lambda \) and \( G \) are not necessarily constants in the scale-covariant theory. A semi-colon denotes covariant derivative and \( \beta_a \) denotes the ordinary derivative with respect to \( x^a \).

A peculiar feature of the theory is that no independent equation for \( \beta \) exists. The usual procedure adopted is to assume a power law of the form (Canuto et al., 1979; and references quoted therein)

\[ \beta(t) = \left( t_0/t \right)^\varepsilon, \quad \varepsilon = \pm \frac{1}{2}; \pm 1, \quad t_0 = \text{constant}, \]

and then to try to determine which one fits most closely the observations. However, there are certain situations in which we can determine specific forms the gauge function and one of these is revealed by a study of the static vacuum Friedmann models.

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2. Field Equations and Solutions

By homogeneity, we regard $\beta$ and $\Lambda$ as being functions only of the time. For the Robertson–Walker metric
\[ ds^2 = -dt^2 + R^2(t) \left[ dr^2/(1 - kr^2) + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \]
where $k = \pm 1$ or 0, the vacuum field equations are
\begin{align*}
3\ddot{R}/R + 3\dot{\beta}/\beta + 3\dot{R}/(R\beta) - 3\dot{\beta}^2/\beta^2 &= \Lambda, \quad (4) \\
3(\dot{R}/R + \dot{\beta}/\beta)^2 + 3k/R^2 &= \Lambda; \quad (5)
\end{align*}
where the overhead dot denotes a derivative with respect to the time. Since we have three unknowns $R$, $\beta$, and $\Lambda$ in Equations (4) and (5), we may solve the system for the static case. Setting $\dot{R} = \dot{\dot{R}} = 0$ in the above equations, we obtain
\begin{align*}
3\ddot{\beta}/\beta - 3\dot{\beta}^2/\beta^2 &= \Lambda, \quad (6) \\
3\dot{\beta}^2/\beta^2 + 3k/R_0^2 &= \Lambda, \quad (7)
\end{align*}
where $R_0 = \text{constant}$.

We now present our solutions for the various values of $k$.

$k = 0$:
\[ \Lambda = A/t^2, \quad \beta = D/t, \quad (8) \]
where $A$ and $D$ are constants.

$k = +1$:
\[ \Lambda = \frac{12}{R_0^2} + \frac{12}{R_0^2} \frac{\tan(2t/R_0) - 1}{1 + \tan(2t/R_0)}, \quad \beta = MA^{1/2}, \]
where $M = \text{constant}$ and where we have chosen a constant of integration to be zero so that $\Lambda(0) = 0$.

$k = -1$:
\[ \Lambda = \frac{12}{R_0^2} - \frac{12}{R_0^2} \frac{[1 + \exp(4t/R_0 + P)]^2}{[1 - \exp(4t/R_0 + P)]^2}, \quad \beta = QA^{1/2}, \]
where $P$ and $Q$ are constants.

3. Discussion

We note that the form $\beta \sim 1/t$ which we have found in the solution (8) is one of the forms that was previously considered as being possibly consistent with observations (cf. Equation (3)). We thus believe that a study of static vacuum solutions in the scale-covariant theory could possibly lead to further interesting forms for the gauge function, which could be checked with observational results. Finally we remark that the solutions presented here do not have any general relativistic analogue.