EFFECT OF OBLATENESS ON TRIANGULAR SOLUTIONS AT CRITICAL MASS

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Abstract. At critical mass the triangular equilibria in the planar restricted three-body problem, when the more massive primary is an oblate spheroid with its equatorial plane coincident with the plane of motion, are in general unstable due to the presence of secular terms in the solutions of linearized equations of motion in the vicinity of these points. Existence of retrograde elliptic periodic orbits is established through suitable velocity components. The eccentricity of these orbits increases with the oblateness.

1. Introduction

It is well known that the triangular solutions of the restricted three-body problem are stable if the mass parameter $\mu$ is smaller than the critical mass value, $\mu_0 = 0.038520\ldots$. For the case where the more massive primary is considered as an oblate spheroid with its equatorial plane coincident with the plane of motion, Subba Rao and Sharma (1975) established that the range of the mass parameter, giving rise to stable triangular solutions, decreases with the increase in the oblateness. Herein the problem has been further studied to examine the solutions of the linearized equations of motion around the triangular points for $\mu_{\text{critical}} = \mu_0 - (1 + 13/\sqrt{69})A_1/9$, $A_1$ being the oblateness coefficient of the larger primary. The solution is in general unstable due to the presence of secular terms coming through the double roots of the characteristic equation. Suitable choice of velocity components eliminates the secular terms and the solution represents retrograde elliptic periodic motions. Unlike the classical case, the characteristic roots and eccentricity are found to be functions of the oblateness coefficient. It is further noted that the eccentricity of the periodic orbits increases with oblateness. Part of this study appears in the doctoral thesis of R. K. Sharma.

2. Equations of Motion

The equations of motion in the dimensionless barycentric synodic coordinate system $(x, y)$ are

$$\ddot{x} - 2n\dot{y} = \frac{\partial \Omega}{\partial x},$$

$$\ddot{y} + 2n\dot{x} = \frac{\partial \Omega}{\partial y},$$

where $n$ is the mean motion of the primary and $\Omega$ is the mean motion function.
\[
\Omega = \frac{n^2}{2} \left[ (1 - \mu)r_1^2 + \mu r_2^2 \right] + \left( \frac{1 - \mu}{r_1} \right) + \frac{\mu}{r_2} + \frac{A_1(1 - \mu)}{2r_1^3},
\]

\[
n^2 = 1 + \frac{3}{2}A_1.
\]

The location of the triangular equilibrium points, given by

\[
\frac{\partial \Omega}{\partial x} = \frac{\partial \Omega}{\partial y} = 0, \quad (y \neq 0)
\]

results in

\[
r_1 = 1, \quad r_2 = 1/(1 + \frac{3}{2}A_1) \leq 1.
\]

Following the notations and terminology of Szebehely (1967), the first variational equations near \(L_4,5(a, \pm b)\) can be written as

\[
\begin{align*}
\ddot{x} - 2n\dot{y} &= \Omega_{xx}\dot{x} + \Omega_{xy}\dot{y}, \\
\ddot{y} + 2n\ddot{x} &= \Omega_{xy}\dot{x} + \Omega_{yy}\dot{y}. \\
\end{align*}
\]

The characteristic equation of formulas (1) is

\[
\lambda^4 + (4n^2 - \Omega_{xx} - \Omega_{yy})\lambda^2 + \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 = 0,
\]

whose roots are given by

\[
\lambda_1 = \frac{1}{2} \pm \frac{1}{2}, \quad \lambda_2 = -\frac{1}{2}, \quad \lambda_3 = -\frac{1}{2}, \quad \lambda_4 = -\frac{1}{2},
\]

where

\[
\Lambda_{1,2} = -\frac{1}{2}(4n^2 - \Omega_{xx} - \Omega_{yy}) \pm \frac{1}{2}D^{1/2},
\]

with

\[
D = 4n^2 - \Omega_{xx} - \Omega_{yy} - 4(\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2).
\]

For \(D > 0\), the characteristic roots in (2) are pure imaginary and the solution of Equations (1) is stable containing long- and short-periodic terms, representing retrograde periodic elliptic motion (Sharma and Rao, 1976), inclined at an angle

\[
\alpha = \frac{1}{2} \tan^{-1} \left[ 2\Omega_{xy} / (\Omega_{xx} - \Omega_{yy}) \right],
\]

to the x-axis.

For \(D < 0\), the characteristic roots are complex with non-zero real parts and the solution is unstable.

For \(D = 0\), the roots

\[
\lambda_1,3 = ik^{1/2}, \quad \lambda_2,4 = -ik^{1/2},
\]

\[
k = (4n^2 - \Omega_{xx} - \Omega_{yy})/2 > 0,
\]

are purely imaginary and equal in pairs. It will be seen that in this case the solution is unstable, due to the presence of secular terms in \(t\). However, if the initial coordinates