AN IMPROVED AND GENERALIZED THEORY FOR THE
COLLISIONAL EVOLUTION OF KEPLERIAN SYSTEMS

K. A. HÄMEEN-ANTTILA
Aarne Karjalainen Observatory, University of Oulu, Finland

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Abstract. A calculation of collisional integrals with a higher accuracy yields excellent agreement between computer simulations and the collisional theory of Keplerian systems. Inclusion of axial rotation of particles modifies the evolution but does not introduce qualitatively new phenomena. Friction between the particles has a stabilizing influence, while deviations from an exactly spherical shape produce an opposite effect. The rotation of spherical or irregular bodies cannot prevent a final flattening of the system into a monolayer without also causing its disintegration. Computer simulations with a small number of particles do not represent the typical collisional evolution. They provide a test for the theory, but may sometimes lead to a misinterpretation of astronomical phenomena.

1. Introduction

Recent computer simulations of Keplerian systems (Lukkari, 1978) confirm fairly well the predictions of the statistical theory of collisions in its revised form (Hämeen-Anttila, 1976, 1978) provided that the initial configuration does not violate certain theoretically predictable restrictions (Hämeen-Anttila, 1978). Owing to this agreement, an elimination of some unnecessary approximations in the construction of the theory was considered worth trying. The most obvious one is the calculation of the integrals which appear in various collisional averages and in the frequency of collisions. The integrand includes the relative velocity of the particles, which was replaced by an adequate mean value in the original theory. This produces errors of ~30%. Another defect is the treatment of friction, which was taken into account as an undetermined coefficient $l$ in the distribution of post-collisional velocities.

The integrals are now calculated with a higher precision, and the influence of friction is derived from a more accurate analysis of the collisions. This implies a generalization of the theory for rotating particles. These are treated as spheres when the energy and angular momentum of the axial rotation are calculated, but a generalization in the concept of friction also permits a qualitatively correct treatment of irregular bodies.

The finite size of the particles is taken into account in the construction of theory. However, after having seen its contribution to the final equations, we shall use a simpler approximation in which the ratio of the particle radius to the distance from the central body is assumed to be much smaller than the typical values of the orbital eccentricities and inclinations.
No effort is made to eliminate the restriction which is introduced by the upper bound of the third derivative in the initial distributions of density, eccentricity and inclination (Hämeen-Anttila, 1978; Lukkari, 1978). The computer simulations invariably show that a violation of this restriction leads to a rapid expansion which reduces the anomalous term of the flux vector. In astronomical systems we may always assume that the final small value has already been reached. In computer simulations the third derivative ought to be studied before they are applied as tests for theoretical predictions or as a means of interpreting astronomical phenomena. The limits of validity are more precisely discussed in Section 10.

The system is assumed to consist of identical particles which have small orbital eccentricities and inclinations. The basic equations are derived directly from Boltzmann's equation. An approximative distribution function for orbital eccentricities and inclinations is theoretically constructed to calculate the collisional averages, but it is found to agree with the semi-empirical distribution function used in the original theory.

2. Orbital Elements

The same orbital elements which were used in the original theory (Hämeen-Anttila, 1976, 1978) also provide an adequate basis for a more accurate theory. The first is

$$\phi = 2\gamma m/r - \dot{r}^2,$$

where $\gamma$ denotes Newton's constant, $m$ the mass of the central body, $r$ the radial coordinate, and $\dot{r}$ the velocity vector. For a particle of mass $\mu$ the total energy is obviously $-\mu\phi/2$. Two other orbital elements are given by the vector $e$ which represents the projection of the perihelion vector in the $xy$-plane,

$$e = \{(N/\gamma m) \times \left[ \vec{r} \times (r \times \dot{r}) - \gamma m \vec{r}/r \right] \} \times N,$$

the vector $N$ being a unit vector along the $z$-axis. The components of the vector

$$l = (N/\gamma m) \times (r \times \dot{r}),$$

which points to the ascending node, correspond to the fourth and fifth element. The sixth orbital element is the perihelion time $t_0$.

The quantities $\phi$, $e$ and $l$ can also be written as

$$\phi = \gamma m/a, \quad e = e(N \times a) \times N/a, \quad l = N \times (a \times b)/\sqrt{\gamma ma^2},$$

where $e$ denotes eccentricity, $a$ the vectorial semi-major axis ($|a| = a$) and $b$ the vectorial semi-minor axis ($|b| = a\sqrt{1 - e^2}$, $a \cdot b = 0$). For small values of $e$ and $i$, we have $|e| \simeq e$ and $|l|\sqrt{\phi} \simeq i$. The expressions of $e$ and $l$ in terms of $e$, $i$ and of the lengths of perihelion and of ascending node have been given elsewhere (Hämeen-Anttila, 1975).