II: The Asymptotic Effect of Collective Excitations on the Cosmological Expansion of the Homogeneous and Isotropic, on the Average, Universe

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Abstract. At the stage of a weak turbulence the interactions between excitations are negligible, and potential, vortical and gravitational perturbations may be considered independent. In this approximation the analytical solutions to exact equations of the turbulence theory are found. It is shown that only such excitations whose effective equations of state do not coincide with those of matter do contribute to the deviation from the Friedmann law. At the stage \( p = \frac{1}{39} \) this involves potential and vortical turbulences. A model of the Universe from gravitational waves is constructed. The influence of the turbulence on the course of the expansion is essential till the beginning of, and probably during, the synthesis of light elements. The rate of cosmological expansion and gravitational instability decreases if the potential turbulence predominates over the vortical one, and increases in the opposite case.

The present paper is the second part of the paper entitled ‘Turbulence in Cosmology’, and will be concerned with a treatment of the case of weak turbulence by means of exact equations set up in Paper I. The numeration of sections and equations in the two papers is continuous.

4. Equations of the Quasi-Linear Theory

The quasi-linear theory of turbulence neglects the interactions between turbulence pulsations. In relativistic cosmology such an approximation is justified asymptotically for large values of the radius of curvature when the turbulence field energy is less than the rest energy

\[ \xi_2 \ll 1. \tag{4.1} \]

According to the parameter (4.1), we may confine ourselves solely to such terms of Equations (2.18), (2.19) which are quadratic in the perturbation amplitudes, and to the linear ones in Equations (2.20), (2.21). It should be noted that for Equations (2.20), (2.21) the inequality (4.1) permits to neglect the difference between the square of the turbulence amplitude and its mean as compared with the linear terms. For neglecting every member of the difference a more rigid condition \( \hbar/\lambda_2^2 \lesssim 1 \) is needed.

In the most convenient 4-dimensional notation the equations of the quasi-linear theory may be written as

\[
\begin{align*}
R_i^i - \frac{1}{2} \delta_i^j \tilde{R} &= \kappa \xi_i^i + \kappa \left\langle 3 \xi_i^i \right\rangle - \left\langle 2 \mathcal{P}_i^i \right\rangle + \frac{1}{2} \delta_i^j \left\langle 2 \mathcal{P} \right\rangle + \\
&+ \frac{1}{2} \left\langle \mathcal{H}^1 (\mathcal{P}^m - h_m^m \tilde{R}_e^e) \right\rangle - \frac{1}{2} \delta_i^j \left\langle \mathcal{H}^2 (1 \xi_m^m - h_m^m \tilde{R}_e^e) \right\rangle,
\end{align*}
\tag{4.2}
\]

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where \( \mathcal{P}_k \), \( \mathcal{P}_q \) are the linear, and \( \mathcal{P}_k \), \( \mathcal{P}_q \) the quadratic expressions in the perturbation amplitudes, obtained from Equations (2.12), (2.15).

The limits of applicability of Equations (4.2)-(4.5) are set up by the inequality (4.1). However, it should be noted that the perturbations of a free gravitational field (gravitational waves) constitute an exception. For \( \tau_i^k = 0 \) and \( \lambda^k_\xi \leq 1 \) the terms of the highest order are already contained in the linear part of (2.20) and in the quadratic part of (2.18) (Isaacson, 1968). Therefore, for gravitational waves \( \xi_\xi \sim 1 \) lies within the limits of applicability of the quasi-linear theory equations. This circumstance allows us to investigate to a sufficiently full extent the cosmological role of gravitational waves on the basis of Equations (4.2), (4.3) (see Section 7.2 of this paper).

Let us find the equations of the quasi-linear theory in a 3 + 1-dimensional form for a homogeneous and isotropic, on the average, Universe. The form of the interval (3.1) leads to a necessary condition

\[ h_0^5 = 0. \]  

(4.6)

This condition is a consequence of the self-consistency of Equations (4.2)-(4.7). All terms in (4.2) must be functions of the cosmological time, but the condition (4.6) means that the proper time of the reference system in which the evolution of turbulence fields is investigated coincides with the cosmological time. From Equation (4.6), the condition \( \tilde{u}_\xi = 0 \) (following from (3.2)) and Equation (4.5), \( \nu_\xi = 0 \) is also obtained.

On passing to the 3 + 1-dimensional form of Equations (4.3)-(4.4) we shall introduce the operations of the 3-dimensional covariant differentiation over the space with the metric \( \gamma_{ab} = a^2 \delta_{ab} \) and 3-dimensional quantities \( (3)h^a, (3)h^a, (3)\mathcal{V}_a, (3)h^a = -h^a, (3)\mathcal{V}_a = -v_a \).

Hereafter we shall deal only with the 3-dimensional quantities and the index (3) will be dropped in the corresponding tensors.

The condition of applicability of the statistical description (2.23) in the case investigated takes the form

\[ \frac{1}{K^2_T} \ll 1, \quad \left( \frac{\dot{a}}{a} \right)^2 \frac{a^2}{K^2_T} \ll 1, \quad \frac{\dot{a}}{a} \frac{a^2}{K^2_T} \ll 1, \]  

(4.7)

where \( K_T = a/\lambda_T, \) \( \dot{a} = da/dt \).

According to the parameters (4.7) only these terms will be left on the right-hand side