A COMPARATIVE STUDY OF BRANS-DICKE AND GENERAL RELATIVISTIC COSMOLOGIES IN TERMS OF OBSERVATIONALLY MEASURABLE QUANTITIES

RONNIE C. BARNES

and

ROBERT PRONDZINSKI

Physics Department, University of Missouri, Columbia, Mo., U.S.A.

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Abstract. A comparison between general relativistic and Brans-Dicke cosmologies is made in terms of quantities measurable by an observational astronomer. Numerical integration of the Brans-Dicke field equations was employed to find the relationships of the mean density of cosmic matter, the age, and the time derivative of the gravitational constant to the Hubble constant and deceleration parameter. The difference between general relativistic and Brans-Dicke apparent magnitude-redshift diagrams was found to be negligible even at large redshifts under the assumption of no galactic evolution in absolute magnitude.

1. Introduction

In 1961 Brans and Dicke proposed a new scalar-tensor theory of gravitation based in part upon considerations of Mach's principle. One of the first problems studied by those authors was that of the expansion rate of a homogeneous model universe under the requirement of zero-pressure. They showed that the expansion parameters for the flat space general relativistic GR and Brans-Dicke BD models were about the same provided plausible values for the coupling constant $\omega$ in their theory were chosen. Hence, they concluded "that it would be difficult to distinguish between the two theories on the basis of space geometry alone". However, no quantitative estimate of how small these differences were was given in terms of quantities actually measurable by the observational astronomer such as apparent magnitude and redshift.

Additional considerations point to the need for a more detailed comparative study of BD and GR model universes. First, the observational astronomer can measure the present values of the Hubble parameter $H_0$ and the deceleration parameter $q_0$. As Sandage (1961) and others have shown, knowledge of $H_0$ and $q_0$ is sufficient for determining the age $t_0$ of any homogeneous zero-pressure GR model universe having a zero cosmical constant. Because BD theory reduces to GR as $\omega$ approaches infinity one should expect a similar, if not identical, relation between $t_0-H_0-q_0$ to exist in BD cosmology. However, due to the absence of a sufficiently detailed study of BD model universes the exact behavior of this relation has remained unknown.

Second, one should anticipate a relation between the present rate of change of the gravitational constant $G_0$ and $q_0$ and $H_0$ in Brans-Dicke cosmology for the following reason. In GR $q_0$ can be expressed as a function of the present mean density of the universe $\rho_0$ and $H_0$. Again, due to the formal similarity between GR and BD theory,
a $q_0 - H_0$ relation would be expected in BD cosmologies. However, in BD theory the time derivative of the scalar field $\phi$ is directly proportional to $\varphi$ (see Equations (3) and (4) of Section 2). Therefore, one intuitively expects a connection between $q_0$ and $\dot{\varphi}_0$ or equivalently between $q_0$ and $\dot{G}_0$ since $\frac{\dot{G}}{\dot{G}_0} = \frac{4 + 2\omega}{4 + 3\omega} G$.

In order to find the behavior of the relations discussed above, as well as the difference between GR and BD magnitude-redshift diagrams, we constructed a grid of BD model universes. Section 2 contains a brief outline of the procedure used to generate these models and their magnitude-redshift diagrams. Sections 3 and 4 contain a discussion of the results and conclusions which may be drawn from our models.

### 2. Techniques

The starting point of any investigation dealing with a homogeneous and isotropic model universe is the Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \lambda r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (1)$$

where $a(t)$ the scale factor is to be determined from the field equations. Here $\lambda$ is the constant of space curvature and $r, \theta, \phi$ a set of co-moving spherical coordinates. With this metric the BD field equations for a homogeneous, zero-pressure model universe become (Brans-Dicke 1961)

$$\left( \frac{a}{\dot{a}} + \frac{1}{2} \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\lambda}{a^2} = \frac{1}{4} \left( 1 + \frac{2}{3} \omega \right) \left( \frac{\dot{\phi}}{\phi} \right)^2 + \left( 1 + \frac{2}{3} \omega \right) \frac{\dot{\phi}}{\phi} \frac{1}{t}, \quad (2)$$

$$\dot{\phi} a^3 = \frac{8\pi Q}{3 + 2\omega} t, \quad (3)$$

$$Q = q a^3 = \text{constant.} \quad (4)$$

Equations (2) and (3) were numerically integrated on the University of Missouri's IBM 360/365 using the Runge-Kutta method. Starting values for $a$ and $\phi$ were generated from the known analytic flat space solution since for sufficiently small $t$ the curved space solution differs only infinitesmally from the flat space solution (Brans-Dicke, 1961). However, while in principle the value of $Q$ is arbitrary, a trial integration of Equations (2) and (3) will show that the resulting curved space model may have an unrealistic value of $q_0$ if a poor choice is made.

The following method was used to pick 'reasonable' $Q$-values. First an expression for $q_0$ in terms of $H_0$, $G_0$, and $\omega$ was derived for the case $\lambda = 0$. If Equations (29a), (60) and (61) of Brans and Dicke (1961) are substituted into Equations (2) above, we obtain

$$\frac{8\pi q_0}{3H_0^2} = \frac{(4 + 2\omega)(4 + 3\omega)}{6(1 + \omega)^2} \frac{1}{G_0} \quad (\lambda = 0), \quad (5)$$

which fixes $q_0$ once $\omega$, $H_0$, and $G_0$ are given. Next a value for $a_0$ was chosen and the value of $Q$ was computed from Equation (4). It was found by trial and error that if $a_0$