TIDAL EVOLUTION IN CLOSE BINARY SYSTEMS, II*

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Abstract. The aim of the present investigation will be to determine the explicit forms of differential equations which govern secular perturbations of the orbital elements of close binary systems in the plane of the orbit (i.e., of the semi-major axis $A$, eccentricity $e$, and longitude of the periastron $\omega$), arising from the lag of dynamical tides due to viscosity of stellar material. The results obtained are exact for any value of orbital eccentricity comprised between $0 \leq e < 1$; and include the effects produced by the second, third and fourth-harmonic dynamical tides, as well as by axial rotation with arbitrary inclination of the equator to the orbital plane.

In Section 2 following brief introductory remarks the variational equations of the problem of plane motion will be set up in terms of the rectangular components $R$, $S$, $W$ of disturbing accelerations with respect to a revolving system of coordinates. The explicit form of these coefficients will be established in Section 3 to the degree of accuracy to which squares and higher powers of quantities of the order of superficial distortion can be ignored. Section 4 will be devoted to a derivation of the explicit form of the variational equations for the case of a perturbing function arising from axial rotation; and in Section 5 we shall derive variational equations which govern the perturbation of orbital elements caused by lagging dynamical tides.

Numerical integrations of these equations, which govern the tidal evolution of close binary systems prompted by viscous friction at constant mass, are being postponed for subsequent investigations.

1. Introduction

In a previous paper of this series (Kopal, 1972b; hereafter referred to as Paper I), we outlined the evolutionary processes which produce secular changes in the dimensions of the orbit of close binary systems of constant angular momentum; and established that – for systems exhibiting no mass loss or exchange – the principal source of such evolution are dynamical tides lagging in phase behind the variations in external field of force on account of the viscosity of stellar material. In Section 5 of Paper I we considered, moreover, in some detail the mechanics of a secular increase in the semi-major axis $A$ of the relative orbit of such systems, due to an exchange between the rotational and orbital momenta caused by tidal friction in systems whose components do not rotate in synchronism with their revolution; and determined the magnitude of the corresponding tidal lag from the required constancy of the total momentum of the system.

In doing so we restricted, however, the form of the orbit to remain circular; and this restriction – advantageous as it may be from purely mathematical point of view – cannot really be justified on physical grounds. For the same forces which cause $A$ to...
vary are bound to exert a similar influence also on the orbital eccentricity \( e \); and its variations turn out to be coupled with those of \( A \), thus calling for a simultaneous approach to their solution. Moreover, if the equatorial planes of the components are inclined to that of their orbit, perturbations in \( A \) or \( e \) prove to depend also on the longitude of the periastron \( \omega \) which represents the third element of the Keplerian orbits in a plane.

In the present paper we propose, therefore, to set up a simultaneous system of variational equations for the three elements \( A, e, \) and \( \omega \) of Keplerian orbits, governing variations of these elements which are produced by inclined axes of rotation and (or) lagging dynamical tides. Simultaneous solutions of such equations for appropriate sets of boundary conditions should then disclose a complete march of tidal evolution which unfolds in close binary systems consisting of detached components with secularly constant mass. Perturbations arising from mass changes (be it exchange or loss) are wholly outside the scope of the present investigation, and will be dealt with in subsequent papers of this series.

2. Equations of the Problem

Let us consider a close binary system consisting of two components of masses \( m_1, 2 \), and adopt the center of gravity of one (say, \( m_1 \)) of them as the origin of inertial system of rectangular coordinates \( x, y, z \), fixed in space, and describing the position of the secondary of mass \( m_2 \) in its relative orbit with respects to \( m_1 \). If so, the differential equations governing the motion of mass \( m_2 \) in space are known to be of the form

\[
\begin{align*}
\frac{d^2x}{dt^2} + G (m_1 + m_2) \frac{x}{r^3} &= \frac{\partial R_{12}}{\partial x}, \\
\frac{d^2y}{dt^2} + G (m_1 + m_2) \frac{y}{r^3} &= \frac{\partial R_{12}}{\partial y}, \\
\frac{d^2z}{dt^2} + G (m_1 + m_2) \frac{z}{r^3} &= \frac{\partial R_{12}}{\partial z},
\end{align*}
\]

(2.1)

(2.2)

(2.3)

where \( G \) denotes the constant of gravitation;

\[
r^2 = x^2 + y^2 + z^2
\]

(2.4)

is the radius-vector in the relative orbit; and \( R_{12} \) represents the ‘disturbing function’ arising from the departure of both components from spherical symmetry because of axial rotation and mutual tidal action.

Conversely, the position of the secondary component of mass \( m_2 \) in the plane of its relative orbit can be expressed in terms of the (time-varying) elements \( A, e, \omega \) of the Keplerian orbit, satisfying (cf., e.g., Tisserand, 1889; Chapter 27) first-order differential