ADIABATIC PULSATIONS AND CONVECTIVE
INSTABILITY OF UNIFORMLY ROTATING GASEOUS MASSES

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Abstract. Third-order virial equations are used to investigate the oscillations and the stability of the sequence of uniformly rotating compressible Maclaurin spheroids, referred to in an inertial frame. It is seen that in the case of the oscillations belonging to the third harmonics, the frequency spectrum of the Maclaurin sequence referred to in an inertial frame is distinct from the spectrum of the Maclaurin sequence considered stationary in a rotating frame of reference.

Considering the Maclaurin sequence in an inertial frame, the neutral point and the point of onset of dynamical instability (corresponding to the third harmonic deformations) are isolated. They occur for the values of the eccentricity $e = 0.73113$ and $0.96696$, respectively. The neutral point is the analogue of the first point of bifurcation along the Dedekind sequence of ellipsoids and is distinct from the neutral point ($e = 0.89926$) along the Maclaurin sequence considered stationary in a rotating frame; this latter point is the analogue of the first point of bifurcation along the Jacobian sequence. Both the Maclaurin sequences in an inertial frame and in a rotating frame become, however, dynamically unstable for the same eccentricity $e = 0.96696$.

1. Introduction

The analysis by Dirichlet and Dedekind (see Chandrasekhar, 1969, hereafter referred to as EFE) has shown that the problem of rotating configurations clearly distinguishes between two frames of reference; an inertial frame fixed in space and a moving frame whose co-ordinate axes coincide at all times with the principle axes of the equilibrium configuration. Confining our attention to the Maclaurin sequence of spheroids, it follows from Chandrasekhar's analysis of the Riemann ellipsoids (Chandrasekhar, 1965) that, in the case of the oscillations belonging to the second harmonics, the characteristic frequencies of the Maclaurin spheroids, considered stationary in a rotating frame of reference, are identical with those of the 'adjoint' configurations of the Maclaurin spheroids referred to in an inertial frame. In particular, both these 'adjoint' configurations become secularly and dynamically unstable at the same values of eccentricity ($e = 0.81267$ and $e = 0.95289$, respectively). The first of these values locates the point of bifurcation of the Jacobian and the Dedekind ellipsoids from the sequence of Maclaurin spheroids (Lebovitz, 1961; Kochhar and Trehan, 1971, hereafter referred to as Paper I; Kochhar, 1973).

However, in the case of the oscillations belonging to the harmonics higher than two, an equilibrium configuration and its 'adjoint' have different frequency spectra (Chandrasekhar, 1965). The characteristic frequencies of oscillation, belonging to the
third harmonics, of the Maclaurin spheroids have been obtained by Chandrasekhar and Lebovitz (1963a, hereafter referred to as Paper II; see also EFE). They have treated the Maclaurin spheroids as stationary in a rotating frame of reference and have isolated (i) the neutral point \( e = 0.89926 \) which is analogous to the first point of bifurcation along the Jacobian sequence of ellipsoids and (ii) the point \( e = 0.96696 \) at which the configuration becomes overstable to deformations corresponding to the third harmonics. Tassoul and Tassoul (1967) have extended these studies to investigate the adiabatic pulsations of the compressible Maclaurin spheroids.

However, when one wishes to include magnetic fields and differential rotation (which is possible in the inviscid case only when magnetic fields are present) in these studies, it is necessary to consider the configuration in an inertial frame of reference. It is with this aim that we examine here the various modes of oscillation of the compressible Maclaurin spheroids in an inertial frame of reference using the third-order virial equations. The extension of these studies to include magnetic fields and differential rotation will be reported in a forthcoming communication.

### 2. The Equilibrium State and the Third-Order Virial Equations

The equilibrium configuration is an homogeneous compressible spheroidal mass rotating with a uniform angular velocity \( \Omega \). The linear velocity is given by

\[
u_\sigma = (-1)^\sigma \Omega x_{\sigma},
\]

where \( \sigma \) stands for 1 or 2 and \( \sigma \) is 1 if \( \sigma \) is 2 and vice versa. The third-order virial equations referred to in an inertial frame and appropriate to an inviscid gaseous mass of density \( \varrho \) and isotropic pressure \( p \) are

\[
\frac{d}{dt} \int q u_i x_j x_k \, dx = 2T_{ij;k} + 2T_{ik;j} + W_{ij;k} + W_{ik;j} + \delta_{ij} \Pi_k + \delta_{ik} \Pi_j,
\]

where

\[
T_{ij;k} = \frac{1}{2} \int q u_i u_j x_k \, dx,
\]

\[
\Pi_k = \int p x_k \, dx
\]

and

\[
W_{ij;k} = -\frac{1}{2} \int \varphi \varphi_{ij} x_k \, dx.
\]

In Equation (5) \( \varphi_{ij}(x) \) is the tensor potential at \( x \) due to a mass distribution \( \varrho(x') \) at \( x' \), defined by

\[
\varphi_{ij} = G \int q(x) \frac{(x_i - x'_i) (x_j - x'_j)}{|x - x'|^3} \, dx'.
\]