ON THE $\Lambda$-TRANSFORM OF THE EXPONENTIAL INTEGRALS

MOHAMED ADEL SHARAF*
Dept. of Astronomy, University of Cairo, Egypt

(Received 31 July, 1978)

Abstract. In this paper, a technique of recursive analysis is developed for the integral transform $\Lambda$ of the exponential integral functions $E_n$ which is denoted as $\lambda_n(\tau)$. The main result of this analysis enables us to establish a two-term recurrence formula for $\lambda_n(0)$ and a three-term recurrence formula for $\lambda_n(\tau); \tau \neq 0$. A computational algorithm based on these formulae is also constructed and its numerical results for $n = 2(1)25$ are presented to 15-digit accuracy.

1. Introduction

One of the most important and useful tools in the solution as well as in the discussion of many transfer problems are the integral transforms introduced by Hopf (cf., e.g., Kourganoff, 1952; p. 40). Of these, the $\Lambda$ integral transform, defined by

$$\Lambda\{f(t)\} = \frac{1}{2} \int_0^\infty f(t)E_1(|t - \tau|) \, dt,$$

where

$$E_k(t) = \int_1^\infty \frac{e^{-tv}}{v^k} \, dv$$

is the exponential integral function of order $k$. In fact, the $\Lambda$ integral transform is the cornerstone in the iterative improvement for the approximate solutions of all of the standard transfer problems. By a general survey, one can at once deduce that, $E_k$ are the functions which appear naturally in the development of the majority of the transfer problems. Moreover, a linear combination of such functions indicates extremely accurate representation of many functions of the transfer theory (cf., e.g., Sharaf, 1978).

It is natural, therefore, to look to $\lambda_n(\tau)$ the $\Lambda$ integral transform of $E_n$. This is not only for the above reasons but also for an additional one that is; the analysis performed on $\lambda_n(\tau)$ exhibits all the characteristic features of the other integral transforms of $E_n$. Kourganoff gave the form of $\lambda_n(\tau)$ in terms of a linear combination of $E_k$, the highly transcendental functions $M_n(\tau)$ and $N_n(\tau)$ (defined in our analysis as $Q_n^{(+1)}$ and $Q_n^{(-1)}$, respectively). Unfortunately, the computations involved in the evaluation of all $\lambda_n(\tau)$ for $n = 2, 3, \ldots N$ (this is the case in which the functions $\lambda$'s appear) using his form is extremely time- and storage-consuming.

In this paper, we shall construct a recursive computational algorithm based on a two-term recurrence relation for all $\lambda_n(0)$ and on a three-term recurrence relation for

* Now at the Department of Astronomy, University of Manchester, England.
all $\lambda_n(\tau)$; $\tau \neq 0$. In order to do so, a technique of recursive analysis is proposed on some auxiliary functions and discussed in Section 2. While Section 3 is devoted to derive the recurrence relations of $\lambda_n(\tau)$ for $\tau = 0$ and $\tau \neq 0$. In Section 4 we give the computational steps for implementing our formulae on a digital computer. Finally, in Section 5 the numerical results of the algorithm for $n = 2(1)25$ are presented to 15-digit accuracy.

2. Recurrence Relations

Let us introduce the auxiliary functions

$$T_n(\tau, z) = \int_1^{\infty} \frac{e^{-nt}}{t^{\tau}(t + z)} \, dt$$

(2.1)

and

$$Q_\tau^{(\delta)}(\tau) = \int_1^{\infty} \frac{e^{-nt}}{t^{\tau+1}} \ln (t + \delta) \, dt.$$  

(2.2)

From Equation (2.1) we obtain

$$T_{n+1}(\tau, 1) = E_{n+1}(\tau) - T_n(\tau, 1),$$

(2.3)

$$T_{n+1}(\tau, -1) = T_n(\tau, -1) - E_{n+1}(\tau),$$

(2.4)

and

$$\pm z T_{n+1}(0, \pm z) = \frac{1}{n} - T_n(0, \pm z).$$

(2.5)

By writing $T_1(0, z)$ and $T_1(0, -z)$ as

$$T_1(0, z) = \frac{1}{z} \lim_{R \to \infty} \left\{ \int_1^{R} \frac{dt}{t} - \int_1^{R} \frac{dt}{t + z} \right\},$$

and

$$T_1(0, -z) = \frac{1}{z} \lim_{\varepsilon \to 0} \left\{ \int_1^{z-\varepsilon} \frac{dt}{t} + \int_{z+\varepsilon}^{R} \frac{dt}{t - z} - \int_1^{z-\varepsilon} \frac{dt}{t} - \int_{z+\varepsilon}^{R} \frac{dt}{t} \right\},$$

we find that

$$T_1(0, z) = \frac{1}{z} \lim_{R \to \infty} \left\{ \ln \left( \frac{1}{1 + z/R} \right) \right\} + \frac{1}{z} \ln (z + 1)$$

and

$$T_1(0, -z) = \frac{1}{z} \lim_{\varepsilon \to 0} \left\{ \ln \left( \frac{z + \varepsilon}{(1 - z)(\varepsilon - z)} \right) + \ln \left( 1 - \frac{z}{R} \right) \right\}.$$  

Thus,  

$$T_1(0, z) = \frac{1}{z} \ln (z + 1)$$  

(2.6)