ON H-FUNCTIONS OF RADIATIVE TRANSFER

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Abstract. Chandrasekhar's $H$-function $H(z)$ corresponding to the dispersion function $T(z) = |\delta_{rs} - f_{rs}(z)|$, where $[f_{rs}(z)]$ is of rank 1, is obtained in terms of a Cauchy integral whose density function $Q(x, \omega_1, \omega_2, \ldots)$ can be approximated by approximating polynomials (uniformly converging to $Q(x)$) having their coefficients expressed as known functions of the parameters $\omega_r$'s. A closed form approximation of $H(z)$ to a sufficiently high degree of accuracy is then readily available by term by term integration.

Solutions of many problems of radiative transfer or of neutron diffusion in infinite media can be obtained in terms of Chandrasekhar's $H$-functions which satisfy the well-known relation

$$H(z)H(-z) = 1/T(z) \quad \text{on} \quad [-\gamma, \gamma]$$

in the complex $z$-plane. (1)

In this paper we consider a particular class of the dispersion function

$$T(z) = |\delta_{rs} - 2z^2 \int_0^{\lambda_r} Y_{rs}(x) \, dx/(z^2 - x^2)|.$$

$[Y_{rs}(x)]$ is of rank 1, so that $T(z)$ reduces to the form

$$T(z) = 1 - 2z^2 \sum_{r=1}^{n} \int_0^{\lambda_r} Y_r(x) \, dx/(z^2 - x^2), \quad n \text{ a positive integer},$$

$$0 < \lambda_n < \lambda_{n-1} < \cdots < \lambda_1 < 1.$$  (2)

$Y_r(x)$ is a real non-negative continuous function of $x$ on $[-\lambda_r, \lambda_r]$, $r=1, 2, 3, \ldots, n$, and satisfies the following conditions:

(i) $Y_r(x)$ fulfills the Hölder condition $\mathcal{H}(\mu)$ on a finite real closed segment $R^* = [-L, L] \ni [-\lambda_r, \lambda_r]$ for every $r$, $Y_1(\lambda_1 + 0) \neq 0$,

$$|Y_r(x) - Y_r(x')| \leq A|x - x'|^\mu, \quad 0 < \mu \leq 1, x, x' \in R^*,$$

$$A \text{ a positive constant}, \ldots$$  (4)

(ii) $\eta_r = \int_0^{\lambda_r} Y_r(x) \, dx \leq \omega_r/2,$  (5)
where
\[ \sum_{1}^{n} \omega_r = 1, \quad \omega_r's \text{ being } \text{positive constants}, \] (5a)

so that
\[ 0 \leq \zeta = 1 - 2 \sum_{1}^{n} \eta_r < 1. \] (5b)

The purpose of this paper is to obtain the H-functions \( H(z) \) in a simple explicit closed form particularly suitable for quick numerical calculations to a sufficiently high degree of accuracy. To achieve our objective we introduce two other similar functions

\[ L_+(z) = \int_{0}^{\lambda} Y_{\eta}(x) \, dx/(z - x), \]

\[ L_-(z) = L_+(-z), \]

so that \( L(z) \) reduces to

\[ L_+(z) = 1 - \sum_{1}^{n} 2z \int_{0}^{\lambda} Y_{\eta}(x) \, dx/(z - x), \quad L_-(z) = L_+(-z). \] (6)

Following Crum (1947) we can easily see (even if \( Y_{\eta}(x)'s \) are simply continuous and do not satisfy the Hölder condition) that (i) when \( (V_{r})Y_{\eta}(x) > 0 \) on \([0, \lambda]\) and \( \zeta > 0 \) \( T(z) \) vanishes only at the simple zeros \( K, -K \) on the finite real axis outside the cut \([-\lambda_1, \lambda_1] \), \( K > \lambda_1 > 0 \), \( L_+(z) \) vanishes only at the simple zero \( j \) on the finite real positive axis outside the cut \([0, \lambda_1] \) and \( L_-(z) \) vanishes only at the simple zero \( -j \) on the finite real negative axis outside the cut \([-\lambda_1, 0] \), \( j > 21 > 0 \), (ii) when \( (\forall r) Y_{r}(x) > 0 \) on \([0, \lambda_r] \) and \( \xi = 0 \) \( K \) and \( j \) go to infinity \( (K, j \rightarrow \infty \text{ as } \xi \rightarrow 0) \). We note that \( 2T(z) = L_+(z) + L_-(z) \) on \([-\lambda_1, \lambda_1]^c \). Let us now set

\[ D(z) = (K^2/\xi^2)((j^2 - z^2)T(z))/[(K^2 - z^2)L_+(z)L_-(z)], \] (7)

\( D(z) \) (i) tends to \( 1 + O(z) \) as \( z \) tends to 0 from within \([-\lambda_1, \lambda_1]^c \), (ii) tends to constants as \( z \) tends to \( \pm \lambda_1 \) from within \([-\lambda_1, \lambda_1]^c \), (iii) is regular and does not vanish on \([-\lambda_1, \lambda_1]^c \), (iv) tends to \( V^2 + O(1/z^2) \) as \( z \) tends to \( \infty \), where

\[ V^2 = K^2/(\xi \xi). \] (8)

Following the usual procedure (cf. Das Gupta, 1956, 1957, 1958a, b; Kourganoff, 1952) we finally have after some very lengthy analysis

\[ D(z) = V^2 D_-(z)/D_+(z), \]

where

\[ \log D_+(z) = R(z) = \sum_{1}^{n} \int_{\lambda_{r+1}}^{\lambda_r} Q_r(x) \, dx/(x - z), \] (9)