SOME GEOMETRIC PROPERTIES OF
MAGNETOGEOSTROPHIC FLOWS

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Abstract. By use of the intrinsic representation based on unit vectors pointing along tangent, principal normal and bi-normal it is proved that steady circulation preserving magnetogeostrophic motions are the circular helical motion whose Lorentz surfaces are concentric circular cylinders.

1. Introduction

The nonlinear character of the basic equations of MHD flows has presented enormous mathematical difficulties to find their exact solutions. Consequently, many interesting devices have been introduced to study the general properties of the flow. The introduction of the geometric theory of curves and surfaces in the magnetogeostrophic flow theory is one of such devices. The first astronomical application occurred in 1899 when Bigelow (cf. Ferraro and Plumpton, 1961) suggested that there were magnetic fields in the Sun. Certainly the Earth and several other planets possess magnetic fields; the Sun shows an amazing variety of hydromagnetic phenomena in addition to solar cycle, and hundreds of magnetic stars have been observed; on a grander scale a rotating gas cloud could scarcely collapse to form a star without magnetic fields to remove its angular momentum; and even a black hole can possess both angular momentum and magnetic field. Suryanarayan (1965) has shown that Bernoulli's surfaces exists in hydromagnetic flow when the magnetic field lines are along a fixed direction. This condition on the magnetic field leads directly to the conclusion that the Lorentz force is conservative. But this not true in general (except for force-free fields). Lorentz force plays a key role in magnetogeostrophic flows (in which the Coriolis force is balanced by pressure gradient). Choubey et al. (1985) and others (cf. Marris, 1973; Roberts and Soward, 1978; Suryanarayan, 1965; Singh et al., 1986) have also tried to study the geometric properties of flow without giving due importance of Lorentz force. Recently Singh et al. (1987) have studied the geometry of field lines considering magnetogeostrophic flow.

In this paper, we have discussed the steady circulation-preserving magnetogeostrophic flows whose magnetic field magnitude is constant on each magnetic field line. By employing the technique of Marris and Passman (1969) we have shown that the class of motion under consideration is the circular helical motion whose Lorentz surfaces (the surfaces containing magnetic field lines and the current lines) are concentric circular cylinder.
2. Basic Equations

In a rotating frame the differential equations governing the flow of incompressible and highly conductive fluids are characterized (cf. Roberts and Soward, 1978) by

\[ \text{div} \, \vec{\nu} = 0, \quad (2.1) \]

\[ \rho \frac{D\vec{\nu}}{Dt} = -(2\rho \vec{\Omega} \times \vec{\nu} - \nabla p + \vec{J} \times \vec{B} + \eta \nabla^2 \vec{\nu}), \quad (2.2) \]

\[ \text{curl}(\vec{\nu} \times \vec{B}) = \vec{0}, \quad (2.3) \]

and

\[ \text{div} \, \vec{B} = 0, \quad (2.4) \]

where \( \vec{\nu}, \vec{B}, \vec{\Omega}, \rho, p, \) and \( \eta \) are velocity field, magnetic field, angular velocity vector, density, and fluid pressure, respectively. \( \vec{J} \) (\( = \text{curl} \, \vec{B} \)) is the current density vector. For slow, steady, inviscid motion, the curl of Equation (2.2) gives

\[ 2\rho(\vec{\Omega} \cdot \nabla)\vec{\nu} + \nabla \times (\vec{J} \times \vec{B}) = \vec{0}, \quad (2.5) \]

This allows magnetogeostrophic flows in which the Coriolis force is balanced by pressure gradient and Lorentz force \( (\vec{J} \times \vec{B}) \), and is an extension of the Proudman-Taylor theorem (cf. Roberts and Soward, 1978). Clearly the magnetic field relaxes the constraint imposed by rotation. For slow, steady, inviscid flow in a rotating system, the Proudman-Taylor theorem requires that

\[ (\vec{\Omega} \cdot \nabla)\vec{\nu} = \vec{0}, \quad (2.6) \]

The use of Equation (2.6) in (2.5) gives

\[ \text{curl}(\vec{J} \times \vec{B}) = \vec{0}. \quad (2.7) \]

Equation (2.7) is a necessary and sufficient condition for the magnetogeostrophic flows to be a circulation preserving, for the Lorentz force can be expressed as the gradient of a scalar point function.

We consider the magnetic field magnitude constant along each magnetic field line. Let \( \vec{B} = B\bar{s} \) be the magnetic field, where \( \bar{s} \) is the unit vector tangent to magnetic field lines then

\[ \bar{s} \cdot \text{grad} \, B = \frac{\delta B}{\delta s} = 0. \quad (2.8) \]

Equations (2.4) and (2.8) imply that

\[ \text{div} \, \bar{s} = 0. \quad (2.9) \]

We prove the following proposition concerning these motions.