ON ROTATING ISOTHERMAL CONFIGURATIONS IN THE POST-NEWTONIAN APPROXIMATION OF GENERAL RELATIVITY

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Abstract. The equations which govern the structure of a rotating, truncated isothermal sphere in the post-Newtonian approximation of general relativity are derived and solved numerically. Each model is parameterized by both a rotation and a relativity parameter. The density inside the configurations is tabulated and graphed as a function of both distance from the center and co-latitude. Relativistic gravitational effects are found to pull the models into states which are considerably more centrally condensed than one predicts classically. Rotation tends to flatten the isothermal configurations into oblate spheroids, though for even the largest rotation parameters the degree of flattening is only a few percent. The computed models may be similar to the cores of relativistic star clusters.

1. Introduction

We begin by considering a self-gravitating, isothermal configuration in mechanical equilibrium. Letting the body rotate uniformly about some axis of symmetry, we assume that general relativistic influences on the structure are non-negligible. We also assume that the density distribution of the object has been truncated at some finite radius. Given the angular velocity and the relative importance of relativistic effects (to be measured by two dimensionless parameters, $v$ and $\sigma$), we would like to describe the body's internal structure. Because the problem as stated above is very complex, it is necessary to adopt certain assumptions and approximations in order to make significant progress. The assumptions and approximations are described and to some extent justified in Section 2.

The equations describing a rotating, relativistic isothermal configuration are given in Section 3. Numerical and graphical solutions of these equations are presented in Section 4. These results are also discussed in some detail in Section 4. Section 5 summarizes the applications of this work.

2. Basic Assumptions

In order to solve the equation of hydrostatic equilibrium for an isothermal configuration consistently within the post-Newtonian framework, it is necessary to specify the distribution of angular velocity within the body. Throughout this work the isothermal models are assumed to rotate uniformly (like rigid bodies) with angular velocity $\Omega$.

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The assumption of uniform rotation is far from trivial. Marks (1970) has shown that in viscous stars, any rotation law centered on the axis of rotation will lead to severe stability problems. Thus, any simple rotation law we adopt will be subject to some doubt. The assumption of uniform rotation is simple to handle, and probably realistic enough to indicate the dominant effects of rotation on the isothermal body’s structure.

A second major assumption we make is the permissibility of truncation at a finite radius. Chandrasekhar (1939) has shown that the classical isothermal sphere has both infinite mass and infinite radius. Such configurations do not, of course, exist in nature, but are cut off at some finite radius. Examples include globular star clusters (Wooley and Dickens, 1961; Kormendy and Anand, 1971) and red giant stars with isothermal cores.

Truncation of models is necessary to provide boundary conditions for the isothermal body’s potential function. Ideally, an isothermal model’s density distribution should merge continuously into some other density function which does have a finite radius. Just what the outer density and potential functions are depends on the type of object being studied.

In this work, the object being studied is isothermal out to a certain dimensionless distance $\xi$ from the center; thereafter the density is zero. This step-function density makes the problem solvable at the cost of making the density discontinuous at the chosen value of $\xi$. This discontinuity tends to zero as the boundary is extended to arbitrarily large $\xi$. At $\xi = 10$, the value adopted here, the classical isothermal sphere’s density has dropped to less than 3% of the central density, and the effects of the discontinuity are small enough to be ignored.

3. The Structure Equations and Their Solution

Fahlman and Anand (1971, [8]) have used the post-Newtonian equations of motion and continuity to obtain the post-Newtonian equation of hydrostatic equilibrium for a uniformly rotating object. The equation is

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial p}{\partial r} \right] + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[ \left( 1 - \mu^2 \right) \frac{1}{q} \frac{\partial p}{\partial \mu} \right] - \frac{1}{c^2} \frac{\Gamma_4}{\Gamma_4 - 1} \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \times \right.
\]

\[
\times \left[ r^2 \frac{p}{q^2} \frac{\partial p}{\partial r} \right] + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[ \left( 1 - \mu^2 \right) \frac{p}{q^2} \frac{\partial p}{\partial \mu} \right] \right\} = -4\pi G q +
\]

\[
+ 2\Omega^2 + \frac{1}{c^2} \left\{ 4\Omega^4 r^2 (1 - \mu^2) + 8\Omega^2 (1 - \mu^2) \times \right.
\]

\[
\times \left[ r \frac{\partial}{\partial r} \left( U - D \right) - \mu \frac{\partial}{\partial \mu} \left( U - D \right) \right] -
\]

\[
- 8\Omega^2 \left[ 2D - U \right] - 8\pi G q \left[ U + \left( \frac{3\Gamma_4 - 2}{2\Gamma_4 - 2} \right) \frac{p}{q} \right],
\]