ON THE DISAPPEARANCE OF ISOLATING INTEGRALS
IN DYNAMICAL SYSTEMS WITH
MORE THAN TWO DEGREES OF FREEDOM

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Abstract. We continue to study the number of isolating integrals in dynamical systems with three and four degrees of freedom, using as models the measure preserving mappings $T$ already introduced in preceding papers (Froeschlé, 1973; Froeschlé and Scheidecker, 1973a).

Thus, we use here a new numerical method which enables us to take as indicator of stochasticity the variation with $n$ of the two (respectively three) largest eigenvalues – in absolute magnitude – of the linear tangential mapping $T^{n*}$ of $T^n$. This variation appears to be a very good tool for studying the diffusion process which occurs during the disappearance of the isolating integrals, already shown in a previous paper (Froeschlé, 1971). In the case of systems with three degrees of freedom, we define and give an estimation of the diffusion time, and show that the gambler’s ruin model is an approximation of this diffusion process.

1. Introduction

In a previous paper (Froeschlé, 1971), it has been found, using a four dimensional mapping $T$ as a model problem, that a dynamical system with three degrees of freedom has, in general, either two or zero isolating integrals (beside the usual energy integral).

Let $T$ be a measure preserving mapping of the $(x, y, z, t)$ space over itself defined by

$$
T = \begin{cases}
  x_1 = x_0 + a_1 \sin(x_0 + y_0) + b \sin(x_0 + y_0 + z_0 + t_0) \\
y_1 = x_0 + y_0 \\
z_1 = z_0 + a_2 \sin(z_0 + t_0) + b \sin(x_0 + y_0 + z_0 + t_0) \\
t_1 = z_0 + t_0 
\end{cases} \pmod{2\pi}
$$

(1)

If $b=0$, then this mapping $T$ is the product of two area-preserving mappings $T_1$ of $(x, y)$ on itself and $T_2$ of $(z, t)$ on itself.

The initial conditions $(x_0, y_0, z_0, t_0)$ are taken such that an invariant curve exists for $T_1$ (integrable case) and not for $T_2$ (wild or ‘ergodic’ case).

In this case Froeschlé (1971) has observed that as soon as $b \neq 0$, the value of the isolating integral of $T_1$ is subjected to a kind of random walk. This integral slowly disappears by some diffusion process due to the coupling term $b \sin(x_n + y_n + z_n + t_n)$, which produces a quasi-random perturbation of $(x_n, y_n)$, as the points $(z_n, t_n)$ behave in a quasi-random fashion.

In this article, we study more precisely numerically this diffusion process and some characteristic parameters of an orbit during this diffusion. One of our tools is the variation of the eigenvalues of the linear tangential mapping $T^{n*}$ of $T^n$, which is a good
indicator of the stochasticity of an orbit. In particular, we study the character of C-system (Arnold and Avez, 1967) of $T$ during the diffusion process.

Hence, we look for the number of eigenvalues which grow exponentially. (Froeschlé and Scheidecker, 1973b).

In Section 2, we study the link between linear tangential mappings and the diffusion process. In Section 3, we define and estimate the diffusion time. In Section 4, we study the variation of this diffusion time with the coupling term and with the initial conditions. In Section 5, we study the case of a dynamical system with four degrees of freedom, i.e. a six dimensional mapping.

2. Linear Tangential Mappings and Diffusion Process

In order to study more precisely the dissolution of the isolating integrals of the discrete dynamical system $T$, we give the topology of the mapping $T_1$ in two characteristic cases. This mapping is defined by

$$
T_1 \begin{cases}
  x_1 = x_0 + a \sin(x_0 + y_0) \\
  y_1 = x_0 + y_0.
\end{cases} \pmod{2\pi} \quad (2)
$$

Figure 1 and Figure 2 display typical sets of points for the mapping $T_1$. The initial

Fig. 1. The mapping $T_1$ for $a_1 = -1.3$. 