RELAXATION TIMES IN STRICTLY DISK SYSTEMS

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Abstract. It is shown that the time of relaxation by particle encounters of self-gravitating systems in the plane interacting by $1/r^2$ forces is of the same order of magnitude as the mean orbit time. Therefore such a system does not have a Vlasov limit for large numbers of particles, unless appeal is made to some non-zero thickness of the disk. The relevance of this result to numerical experiments on galactic structure is discussed.

The use of the collisionless Boltzmann equation (Vlasov equation) for investigating galactic structure has been justified by simple estimates of the ratio of the relaxation time by particle encounters to the typical orbit time in the mean field (Chandrasekhar, 1939; Hénon, 1958; Ostriker and Davidsen, 1968). This ratio is of the order of $N/\log N$, where $N$ is the number of particles, so that for large $N$ the effects of particle encounters can be neglected on a mean field time scale.

A galaxy, even though it may be highly flattened, is still a three-dimensional system, and it is to three-dimensional systems that these estimates of relaxation time apply. However, galaxies are often approximated as strictly disk systems in which stars are still assumed to interact by $1/r^2$ forces, but are constrained to move in a plane. Therefore, it is of some interest to consider the problem of relaxation times for strictly disk systems.

A simple order of magnitude estimate of relaxation time may be made as follows. For a system of $N$ particles, each of mass $m$, which is of typical size $R$, the virial theorem gives an estimate of a typical total particle velocity $V$ from the relation

$$V^2 = GN m/R,$$  \(1\)

where $G$ is the gravitational constant. It is useful to distinguish between this total velocity and the typical random velocity $v$ of particles relative to a local frame of rest. This random velocity is some fraction of the total velocity:

$$v = \lambda V, \quad 0 < \lambda < 1.$$  \(2\)

In a collision between two particles with relative velocity $v$ and with impact parameter $b$, the change of velocity is easily estimated to be

$$\Delta v \sim Gm/(bv).$$  \(3\)

This formula holds for $\Delta v \lesssim v$. It breaks down at the point where $\Delta v \gtrsim v$, which is the condition for a close encounter. This occurs when $b \lesssim b_0$, where

$$b_0 = Gm/v^2 \sim R/(\lambda^2 N),$$  \(4\)

using the virial theorem.
The relaxation time $t_R$ may now be estimated as the time required for a typical particle to suffer a close encounter. Since this neglects the cumulative effect of long-range encounters, this will be an overestimate of the true relaxation time. In a time $t_R$ the motion of the particle takes it a distance $vt_R$ relative to neighboring particles. An estimate of the number of close encounters during this time is equal to the surface density of particles

$$ q \sim \frac{N}{R^2} \quad (5) $$

times the area $(vt_R) \cdot (2b_0)$ within which another particle will cause a close encounter with the given particle. Setting this equal to unity leads to the result

$$ t_R \sim \frac{R^2}{(2Nb_0v)}. \quad (6) $$

The orbit time for a typical particle in the mean field is defined by

$$ t_M = \frac{R}{V}. \quad (7) $$

Combining these results yields

$$ \frac{t_R}{t_M} \sim \frac{\lambda}{2}. \quad (8) $$

The relaxation time is seen to be at most the same order of magnitude as the mean orbit time, independent of the number of particles. Therefore the collisionless Boltzmann equation can never be an adequate description of a strictly disk system, however large.

A refined derivation will now be given that includes the cumulative effects of long-range encounters. In this case the relaxation time is defined as the time at which the root mean square velocity change due to encounters is of the same order of magnitude as the typical random velocity $v$. Thus, the condition is

$$ v^2 \sim (\Delta v)^2_{\text{total}} \sim (G^2m^2/v^2)(vt_Rq) \int (2db)/b^2. \quad (9) $$

The quantity $vt_Rq \cdot 2db$ represents the number of particle encounters during the time $t_R$ having impact parameters in the range $b$ to $b+db$. The lower limit of the integral is taken to be $b_0$ where the formula for $\Delta v$ breaks down and the divergence must be cut off. Since the integral converges rapidly for large $b$, the upper limit may be taken as $\infty$. Then

$$ t_R \sim v^3b_0/(2G^3mq)^2 \sim v^3R^3/(2G^2M^2N^2\lambda^2), \quad (10) $$

where the results (4) and (5) have been used. With the virial theorem (1) and Equations (2) and (7) this gives

$$ \frac{t_R}{t_M} \sim \frac{\lambda}{2}. \quad (11) $$

The fact that this estimate is identical with the previous one (8) indicates that the relaxation is substantially due to close encounters, and that the cumulative effect of long-range encounters is of no more than the same order of magnitude.

The relaxation time is seen to be proportional to $\lambda$, implying that the rate of relaxa-