THE EINSTEIN–MAXWELL FIELD EQUATIONS, II

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(Received 3 November, 1980)

Abstract. In order to arrive at more general results solving Einstein-Maxwell's equations our investigation is centered around an electromagnetic spin tensor, which must be chosen in such a way that conservation laws still hold. This notion of the combined tensor is of course closely linked with the unified field equations. We shall avoid in this way the problem of the form of the matter tensor and neglect non-linear gravitational terms in the Ricci tensor. Then, the field equations have as solutions $h_0 = h_0^{(p)} + h_0^{(b)}$, where $h_0^{(p)}$ are particular solutions, which are obtained by direct calculations and $h_0^{(b)}$ are solutions of $\Box h_0^{(b)} = 0$. The quantities $h_0^{(p)}$ are purely electromagnetic in nature, while $h_0^{(b)}$ may represent purely gravitational terms. The results obtained complete the ones which have been published already in the preceeding paper (Dionysiou, 1980a; which will hereafter be referred to as Paper I).

1. Introduction

The Einstein–Maxwell field equations in relativity theory of a system of $n$-charges $e_n$, avoiding in this paper the problem of its internal structure and matter tensor, are defined by (cf. Paper I)

$$R_{ij} = \frac{8\pi G}{c^4} E_{ij}, \quad (1)$$

$$F_{ij} = \frac{4\pi}{c^4} I^n, \quad (2)$$

$F_{ik} + F_{ji} + F_{ij} = 0, \quad (3)$

where

$$F_{ij} = A_{ij} - A_{ji} \quad (4)$$

is the electromagnetic field tensor, which is related to the combined tensor by

$$E_{ij} = T_{ij} - T_{ij}^{(sp)} \quad (5) \quad (Jackson, 1975).$$

Equation (5) can be written explicitly as

$$E_{ij} = \frac{1}{4\pi} \left( -F_{ij} F_{ji} + \frac{1}{4} g_{ij} F_{lm} F^{lm} \right), \quad (6)$$

where

$$T_{ij} = \frac{1}{4\pi} \left( -F_{ij} F^{ij} + \frac{1}{4} g_{ij} F_{lm} F^{lm} \right) - \frac{1}{4\pi} F_{ik} A_{kj} \quad (7)$$

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is the canonical electromagnetic tensor (Jackson, 1975) and

\[ T^{ij(sp)} = \frac{1}{4\pi} (F^{jk} A^i)_k \]  

(8)

may be interpreted as the spin electromagnetic tensor (Belinfante, 1939, 1940; Pauli, 1941; Novak, 1980).

The energy momentum tensor \( E^{ij} \) satisfies the symmetry requirement. Then, we consider a physical isolated system of \( n \)-charges \( e_n \) with the \( E^{ij} \) total energy momentum tensor conserved or expressed mathematically

\[ E_{ij} = 0. \]  

(9)

We can use \( E^{ij} \) to construct the total angular momentum tensor

\[ M^{ijk} = x^i E^{jk} - x^j E^{ik}; \]  

(10)

and because \( E^{ij} \) is conserved and symmetric, \( M^{ijk} \) is also conserved as

\[ M_{ijk} = x^i_{ik} E^{jk} - x^j_{ik} E^{ik} = E^{ij} - E^{ij} = 0. \]  

(11)

The spin tensor \( T^{ij(sp)} \) gives no contribution to the total energy and momentum, but it is important in calculations of the total angular momentum (Jackson, 1975; Novak, 1980).

Equations (2) and (3) are pure Maxwell equations, where

\[ I^i = (c\rho, I) \]  

(12)

is the current four-vector satisfying the continuity equation

\[ I_{ij} = 0. \]  

(13)

The charge density \( \rho \) is defined by

\[ \rho(r, t) = \sum_{\nu=1}^{n} e_{\nu} \delta(r - r_{\nu}(t)), \]  

(14)

where the sum goes over all the charges and \( r_{\nu} \) is the radius vector of the charge \( e_{\nu} \).

The current density vector \( I \) is defined by

\[ I = \rho v, \]  

(15)

where \( v \) is the velocity of the charge at the given point. Lowering the index "\( i \)" in Equation (12) we have

\[ I_i = (c\rho, -I). \]  

(12a)

Also, the four potential vector is defined by

\[ A^i = (\varphi, A); \]  

(16)