Abstract. We define the topology of an (axisymmetric) magnetic field, a set of qualitative labels characterizing the connectivity of the lines of force. Under the special continuous deformations of the $B$-field defined by time evolution under the dynamo equation in the 'convective' regime (where the frozen-in behavior dominates the diffusion), this topology is preserved. This theorem should have applications to the study of time-varying magnetic fields in that regime in the case that exact or even approximate solutions are difficult to obtain. A partial generalization to the general case of convection and diffusion is made. As an application, a critique of Hibberd's recent theory of a time-dependent axisymmetric geomagnetic dynamo whose dipole-like field undergoes successive reversals is given.

1. Introduction

Recently an interesting new idea (Hibberd, 1979) in the theory of the geomagnetic dynamo was proposed. The current density $j$ obeys an Ohm's Law

$$ j = \sigma (E + G \times B) \quad (1.1) $$

in which $G$, dimensionally a velocity field, is proportional to a temperature gradient in the core, so that the emf $G \times B$ is the so-called Nernst emf (Ziman, 1964). $G$ then appears in the place of the usual fluid velocity $v$ in the dynamo equation (Busse, 1978). In addition, everything is assumed axisymmetric; however, the strictures of Cowling's theorem (Cowling, 1934; Busse, 1978) are avoided by assuming a non-steady $B$, in particular, the electric field is then not purely irrotational. In fact, the time-dependence of $B$ is central to the theory, for not only does it predict a dipole-like main field at the earth's surface at any epoch, but, at the same time, a sequence of reversals of the geomagnetic field.

The mechanism proposed is as follows: starting with a predominantly westward current system in the core and associated $B$-field, implying a dipole-like field of the observed polarity at the Earth's surface, this configuration is 'convected' outward toward the mantle by the Nernst 'velocity field' $G$ (assumed radial) in the well-known way of 'frozen-in' fields when the convective term dominates the diffusion term in the dynamo equation (Cowling, 1957). Qualitative arguments are given for the generation at the same time of an eastward current system nested inside the westward one. When the westward system reaches the core-mantle boundary and decays away (since the conductivity $\sigma$ suddenly drops and the diffusion term overwhelms the convective term),
one is left with a predominantly eastward current system in the core and associated
dipole-like geomagnetic field of the opposite polarity. But as this in its turn is convected
outward to the mantle and eventual extinction, another current system, this time
westward, is regenerated inside it. And so on, providing an endless sequence of reversing
dipole-like fields.

However, after many attempts to reproduce this mechanism quantitatively with a
radial Nernst velocity field (temperature gradient) and indeed with any axisymmetric
fluid velocity, we began to entertain doubts as to its possibility. We were then led to
associate a topology with the \( B \)-field (in the sense of the word as used originally in
analysis situs and in the theories of closed surfaces, graphs, etc.). As we define it, this
topology is preserved by the action of the convective dynamo equation, that is, the
dynamo equation minus the diffusion term. But in the mechanism proposed by
Hibberd one topology evolves continuously into a different one. Hence the mechanism
is strictly impossible if the diffusion term can be neglected relative to the convective
term in the core. Even if the diffusion term is retained, we can prove a result, a partial
generalization of the above theory, which is sufficient to rule out Hibberd's mechanism.

We emphasize that this theory is intended to be a first step in treating timevarying
localized systems of magnetic field and current – whether true dynamos or transient –
from a topological viewpoint. If the system is a true dynamo: self-sustaining, quasi-
stationary, etc., then diffusion cannot be neglected (I owe this remark to the referee) and
this theory must be generalized to that case. However, it should apply as it stands to
convective regimes, where the magnetic Reynolds number is very large, such as
interstellar or intergalactic magnetic fields, or possibly even stellar magnetism. The
main advantage of the topological approach, as we see it now, is that it should permit
qualitative predictions of the physical configuration over finite time intervals even in
the case that exact or approximate solutions are out of the question.

The plan of the paper is the following: in Section 2 the topology of the \( B \)-field is
defined. In Section 3 it is proved that the convective dynamo equation preserves this
topology. Section 4 applies this theorem to Hibberd's theory after making precise some
tacit assumptions on the current systems at various epochs. In Section 5 we discuss
what changes in the theory inclusion of the diffusion term brings and prove (less
rigorously) a partial generalization of the theory to that regime.

A stylistic dilemma arises in writing physical papers which, like this one, contain
negative theorems. A certain amount of mathematical care and rigor is unavoidable and
desirable, otherwise they fail in their stated purpose of closing off all loopholes for a
proposed physical theory. Yet a completely rigorous presentation often obscures the
physical ideas. To avoid the repellent appearance of a pure mathematics paper we have
adopted the following compromise. Our various lemmas and theorems are stated and
proved in the somewhat informal mathematical style common in theoretical physics.
Some proofs are omitted, most are only sketched, and many mathematical details are
left out. However, we claim that the rigorous proofs do follow from our explicit
assumptions along the suggested lines of argument by filling in the mathematical
details.